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AN ASYMPTOTIC THEORY FOR DYNAMIC RESPONSE OF DOUBLY CURVED LAMINATED SHELLS

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Abstract—An asymptotic theory for dynamic analysis of doubly curved laminated shells is formulated within the framework of three-dimensional elasticity. Multiple time scales are introduced in the formulation so that the secular terms can be eliminated in obtaining a uniform expansion leading to valid asymptotic solutions. By means of reformulation and asymptotic expansions the basic three-dimensional equations are decomposed into recursive sets of equations that can be integrated in succession. The classical laminated shell theory (CST) is derived as a leading-order approximation to the three-dimensional theory. Modifications to the leading-order approximation are obtained systematically by considering the solvability conditions of the higher-order equations. The essential feature of the theory is that an accurate elasticity solution can be determined hierarchically by solving the CST equations in a consistent way without treating the layers individually. Illustrative examples are given to demonstrate the performance of the theory. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

In the literature three-dimensional analyses of doubly curved laminated shells are scarce. This is mainly due to the inherent complexity of the basic three-dimensional equations in curvilinear coordinates. Recently, a number of approximate elasticity solutions for this type of shells were obtained. The solutions by Bhimaraddi (1991, 1993) and Fan and Zhang (1992) were made possible by dividing the individual layers into sublayers so that the thin shell assumptions may be adopted to approximate the governing equations with *variable* coefficients by ones with *constant* coefficients for each sublayer. This discrete-layer approach is clear in formulation, however, it may not be advantageous in computation because of the necessity of solving a large system of equations resulting from imposing the interfacial continuity conditions. Treatment of the actual laminate system itself, layer by layer, is cumbersome. Artificially dividing the layers further into sublayers is, of course, something to be avoided. Analysis of the problem by solving the partial differential equations with variable coefficients was made by Huang and Tauchert (1992) and Huang (1995) using the Forbenius method. The power series solution is approximate in nature, and the errors are difficult to assess in general.

The displacement-based models in which the formulation is based on certain kinematic assumptions on the through-thickness variations of the displacements, do not treat the system layer by layer. Among them, the classical laminated shell theory (CST), the first-order (FSDT) and higher-order shear deformation theories (HSDT) for doubly curved laminated shells are well known (Reddy, 1984; Reddy and Liu, 1985; Khdeir *et al.*, 1989; Librescu *et al.*, 1989; Leissa and Qatu, 1991). These models provide a simple way for determining the basic structural response of the laminated shells, but are generally inadequate in predicting the interlaminar stresses because of their inability to satisfy the interfacial traction continuity requirement. The validity of the solutions thus obtained are normally checked against the analytic solutions for benchmark problems. Once the solutions are found to be unreliable, improvements cannot be made without entire reformulation. Assessments of various models for multilayered composite shells can be found in Kapania (1989) and Noor and Burton (1990).

In this paper an asymptotic theory for dynamic analysis of doubly curved shells of laminated construction is presented. The laminated shell will be regarded as a heterogeneous shell with nonhomogeneous material properties in the thickness direction. Thus there is no need to consider the system layer by layer nor to treat the interfacial continuity conditions in particular. Studies of the dynamic analysis of nonhomogeneous plates by means of asymptotic expansions were presented long ago by Widera (1970) and Johnson and Widera (1971). In their formulation a straightforward expansion using a single time variable was used and certain assumptions regarding the material compliances were made. The asymptotic equations were derived only up to the second order, and they are in rather complicated forms even in the case of flat homogeneous plates. No numerical results were given to demonstrate the applicability of the theory. In addition, various refined shell theories for the static problems were presented by applying the methods of asymptotic integration (Agalovian, 1966; Logan and Widera, 1980; Widera and Logan, 1980). Asymptotic analysis for dynamic response of laminated shells is not just a matter of applying the standard perturbation method. This will lead to not only equations too cumbersome to be useful but also nonuniform expansions containing secular terms. It will be shown herein that a straightforward expansion using only a single time scale will *not* result in a valid asymptotic solution, whereas uniform expansions can be obtained using the method of multiple scales (Nayfeh, 1981).

This paper is a continuation of the recent study of asymptotic theories of multilayered plates and shells (Tarn, 1994; Tarn and Wang, 1994; Tarn and Yen, 1995). Dynamic analysis of doubly curved laminated shells is formulated on the basis of three-dimensional elasticity. This type of shells is of interest in its own right. Further, by varying the curvature radii, various types of structures, such as the laminated plate (both curvature radii are infinitely large), the spherical shell (curvature radii are equal) and the circular cylindrical shell (one of the curvature radii is infinitely large), are included as special cases. The problem is more complicated to deal with because of the geometries and curvatures involved. Emphasis will be focused on the derivation of a systematic way for determining the higher-order modifications to a uniformly valid asymptotic solution. With the aid of multiple scales, solvability conditions of the higher-order equations are clearly derived in obtaining a uniform expansion free of secular terms. The benchmark free vibration problems will be solved. The natural frequencies obtained according to various models will be compared to demonstrate the performance of the asymptotic analysis.

2. BASIC THREE-DIMENSIONAL EQUATIONS

Consider a doubly curved laminated shell as shown in Fig. 1, in which $2h$ denotes the thickness of the shell. A set of the orthogonal curvilinear coordinates (α, β, ζ) is located on

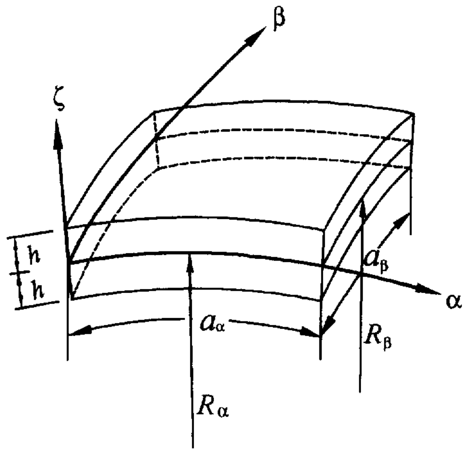

Fig. 1. The geometry and coordinate system for the doubly curved shell.

the middle surface. R_α and R_β are the radii of curvature to the middle surface, and a_α and a_β denote the curvilinear dimensions in the α and β directions, respectively.

The shell of laminate construction is regarded as a nonhomogeneous shell with heterogeneity in the thickness direction. The material is considered to be curvilinear anisotropic, having at each point elastic symmetry with respect to the surfaces ζ = constant. The stress-strain relations of the material are given by

$$\begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\zeta \\ \tau_{\beta\zeta} \\ \tau_{\alpha\zeta} \\ \tau_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\zeta \\ \gamma_{\beta\zeta} \\ \gamma_{\alpha\zeta} \\ \gamma_{\alpha\beta} \end{Bmatrix}, \tag{1}$$

where σ_α, σ_β, σ_ζ, $\tau_{\alpha\zeta}$, $\tau_{\beta\zeta}$, $\tau_{\alpha\beta}$ and ε_α, ε_β, ε_ζ, $\gamma_{\alpha\zeta}$, $\gamma_{\beta\zeta}$, $\gamma_{\alpha\beta}$ are the stress and strain components, respectively. The laminated system is heterogeneous through the thickness such that the elastic constants $c_{ij} = c_{ij}(\zeta)$ are piecewisely continuous functions of ζ.

The kinematic relations in terms of the curvilinear coordinates α, β and ζ can be expressed as

$$\begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\zeta \\ \gamma_{\beta\zeta} \\ \gamma_{\alpha\zeta} \\ \gamma_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} \partial_\alpha/\gamma_\alpha & 0 & 1/\gamma_\alpha R_\alpha \\ 0 & \partial_\beta/\gamma_\beta & 1/\gamma_\beta R_\beta \\ 0 & 0 & \partial_\zeta \\ 0 & \partial_\zeta - 1/\gamma_\beta R_\beta & \partial_\beta/\gamma_\beta \\ \partial_\zeta - 1/\gamma_\alpha R_\alpha & 0 & \partial_\alpha/\gamma_\alpha \\ \partial_\beta/\gamma_\beta & \partial_\alpha/\gamma_\alpha & 0 \end{bmatrix} \begin{Bmatrix} u_\alpha \\ u_\beta \\ u_\zeta \end{Bmatrix}, \tag{2}$$

in which $\partial_\alpha = \partial/\partial\alpha$, $\partial_\beta = \partial/\partial\beta$, $\partial_\zeta = \partial/\partial\zeta$, $\gamma_\alpha = 1+\zeta/R_\alpha$, $\gamma_\beta = 1+\zeta/R_\beta$; u_α, u_β and u_ζ are the displacement components.

The equations of motion are

$$\gamma_\beta \frac{\partial\sigma_\alpha}{\partial\alpha} + \gamma_\alpha \frac{\partial\tau_{\alpha\beta}}{\partial\beta} + \gamma_\alpha\gamma_\beta \frac{\partial\tau_{\alpha\zeta}}{\partial\zeta} + \left(\frac{2}{R_\alpha} + \frac{1}{R_\beta} + \frac{3\zeta}{R_\alpha R_\beta}\right)\tau_{\alpha\zeta} = \gamma_\alpha\gamma_\beta\rho\frac{\partial^2 u_\alpha}{\partial t^2}, \tag{3}$$

$$\gamma_\beta \frac{\partial\tau_{\alpha\beta}}{\partial\alpha} + \gamma_\alpha \frac{\partial\sigma_\beta}{\partial\beta} + \gamma_\alpha\gamma_\beta \frac{\partial\tau_{\beta\zeta}}{\partial\zeta} + \left(\frac{1}{R_\alpha} + \frac{2}{R_\beta} + \frac{3\zeta}{R_\alpha R_\beta}\right)\tau_{\beta\zeta} = \gamma_\alpha\gamma_\beta\rho\frac{\partial^2 u_\beta}{\partial t^2}, \tag{4}$$

$$\gamma_\beta \frac{\partial\tau_{\alpha\zeta}}{\partial\alpha} + \gamma_\alpha \frac{\partial\tau_{\beta\zeta}}{\partial\beta} + \gamma_\alpha\gamma_\beta \frac{\partial\sigma_\zeta}{\partial\zeta} + \left(\frac{1}{R_\alpha} + \frac{1}{R_\beta} + \frac{2\zeta}{R_\alpha R_\beta}\right)\sigma_\zeta - \frac{\gamma_\beta}{R_\alpha}\sigma_\alpha - \frac{\gamma_\alpha}{R_\beta}\sigma_\beta = \gamma_\alpha\gamma_\beta\rho\frac{\partial^2 u_\zeta}{\partial t^2}. \tag{5}$$

We remark that the Lamé parameters are taken to be constants in (2)–(5). This is appropriate for developable shells, but not for undevelopable shells.

The displacements u_α, u_β, u_ζ and transverse stresses $\tau_{\beta\zeta}$, $\tau_{\alpha\zeta}$, σ_ζ will be regarded as the primary field variables. Therefore, let us eliminate the membrane stresses σ_α, σ_β and $\tau_{\alpha\beta}$ from (1)–(5) and express the basic equations in the following form:

$$u_{\zeta,\zeta} = -[l_{31}\ l_{32}]\begin{Bmatrix} u_\alpha \\ u_\beta \end{Bmatrix} - l_{33}u_\zeta + l_{34}\sigma_\zeta, \tag{6}$$

$$\begin{Bmatrix} u_\alpha \\ u_\beta \end{Bmatrix}_{,\zeta} = \begin{bmatrix} l_{11} & 0 \\ 0 & l_{22} \end{bmatrix}\begin{Bmatrix} u_\alpha \\ u_\beta \end{Bmatrix} - \begin{Bmatrix} l_{13} \\ l_{23} \end{Bmatrix} u_\zeta + \begin{bmatrix} l_{14} & l_{15} \\ l_{15} & l_{25} \end{bmatrix}\begin{Bmatrix} \tau_{\alpha\zeta} \\ \tau_{\beta\zeta} \end{Bmatrix}, \tag{7}$$

$$\begin{Bmatrix} \tau_{\alpha\zeta} \\ \tau_{\beta\zeta} \end{Bmatrix}_{,\zeta} = -\begin{bmatrix} l_{41} & l_{42} \\ l_{42} & l_{52} \end{bmatrix}\begin{Bmatrix} u_\alpha \\ u_\beta \end{Bmatrix} - \begin{bmatrix} l_{43} \\ l_{53} \end{bmatrix} u_\zeta - \begin{bmatrix} l_{44} & 0 \\ 0 & l_{55} \end{bmatrix}\begin{Bmatrix} \tau_{\alpha\zeta} \\ \tau_{\beta\zeta} \end{Bmatrix} - \begin{bmatrix} l_{31} \\ l_{32} \end{bmatrix}\sigma_\zeta + \begin{Bmatrix} \rho\dfrac{\partial^2 u_\alpha}{\partial t^2} \\ \rho\dfrac{\partial^2 u_\beta}{\partial t^2} \end{Bmatrix}, \tag{8}$$

$$\sigma_{\zeta,\zeta} = [l_{43}\ l_{53}]\begin{Bmatrix} u_\alpha \\ u_\beta \end{Bmatrix} + l_{63}u_\zeta - [l_{13}\ l_{23}]\begin{Bmatrix} \tau_{\alpha\zeta} \\ \tau_{\beta\zeta} \end{Bmatrix} - l_{66}\sigma_\zeta + \rho\frac{\partial^2 u_\zeta}{\partial t^2}, \tag{9}$$

where

$$l_{31} = \frac{\tilde{c}_{13}}{\gamma_\alpha}\partial_\alpha + \frac{\tilde{c}_{36}}{\gamma_\beta}\partial_\beta, \quad l_{32} = \frac{\tilde{c}_{36}}{\gamma_\alpha}\partial_\alpha + \frac{\tilde{c}_{23}}{\gamma_\beta}\partial_\beta, \quad l_{33} = \frac{\tilde{c}_{13}}{\gamma_\alpha R_\alpha} + \frac{\tilde{c}_{23}}{\gamma_\beta R_\beta},$$

$$l_{34} = \frac{1}{c_{33}}, \quad l_{11} = \frac{1}{\gamma_\alpha R_\alpha}, \quad l_{22} = \frac{1}{\gamma_\beta R_\beta}, \quad l_{13} = \frac{1}{\gamma_\alpha}\partial_\alpha, \quad l_{23} = \frac{1}{\gamma_\beta}\partial_\beta,$$

$$l_{14} = -\frac{c_{44}}{c_{45}^2 - c_{44}c_{55}}, \quad l_{15} = \frac{c_{45}}{c_{45}^2 - c_{44}c_{55}}, \quad l_{25} = -\frac{c_{55}}{c_{45}^2 - c_{44}c_{55}},$$

$$l_{41} = \frac{Q_{11}}{\gamma_\alpha^2}\partial_{\alpha\alpha} + \frac{2Q_{16}}{\gamma_\alpha\gamma_\beta}\partial_{\alpha\beta} + \frac{Q_{66}}{\gamma_\beta^2}\partial_{\beta\beta},$$

$$l_{42} = \frac{Q_{16}}{\gamma_\alpha^2}\partial_{\alpha\alpha} + \frac{(Q_{12}+Q_{66})}{\gamma_\alpha\gamma_\beta}\partial_{\alpha\beta} + \frac{Q_{26}}{\gamma_\beta^2}\partial_{\beta\beta},$$

$$l_{52} = \frac{Q_{66}}{\gamma_\alpha^2}\partial_{\alpha\alpha} + \frac{2Q_{26}}{\gamma_\alpha\gamma_\beta}\partial_{\alpha\beta} + \frac{Q_{22}}{\gamma_\beta^2}\partial_{\alpha\beta},$$

$$l_{43} = \left(\frac{Q_{11}}{\gamma_\alpha^2 R_\alpha} + \frac{Q_{12}}{\gamma_\alpha\gamma_\beta R_\beta}\right)\partial_\alpha + \left(\frac{Q_{16}}{\gamma_\alpha\gamma_\beta R_\alpha} + \frac{Q_{26}}{\gamma_\beta^2 R_\beta}\right)\partial_\beta,$$

$$l_{53} = \left(\frac{Q_{16}}{\gamma_\alpha^2 R_\alpha} + \frac{Q_{26}}{\gamma_\alpha\gamma_\beta R_\beta}\right)\partial_\alpha + \left(\frac{Q_{22}}{\gamma_\beta^2 R_\beta} + \frac{Q_{12}}{\gamma_\alpha\gamma_\beta R_\alpha}\right)\partial_\beta,$$

$$l_{44} = \frac{1}{\gamma_\alpha\gamma_\beta}\left(\frac{2}{R_\alpha} + \frac{1}{R_\beta} + \frac{3\zeta}{R_\alpha R_\beta}\right), \quad l_{55} = \frac{1}{\gamma_\alpha\gamma_\beta}\left(\frac{1}{R_\alpha} + \frac{2}{R_\beta} + \frac{3\zeta}{R_\alpha R_\beta}\right),$$

$$l_{63} = \frac{Q_{11}}{\gamma_\alpha^2 R_\alpha^2} + \frac{2Q_{12}}{\gamma_\alpha\gamma_\beta R_\alpha R_\beta} + \frac{Q_{22}}{\gamma_\beta^2 R_\beta^2}, \quad l_{66} = \frac{1}{\gamma_\alpha\gamma_\beta}\left(\frac{1}{R_\alpha} + \frac{1}{R_\beta} + \frac{2\zeta}{R_\alpha R_\beta}\right) - \frac{\tilde{c}_{13}}{\gamma_\alpha R_\alpha} - \frac{\tilde{c}_{23}}{\gamma_\beta R_\beta},$$

$$Q_{ij} = c_{ij} - \frac{c_{i3}c_{j3}}{c_{33}} \quad (i,j = 1,2,6), \quad \tilde{c}_{ij} = \frac{c_{ij}}{c_{33}}.$$

The membrane stresses can be expressed in terms of the displacements and transverse stresses as

$$\begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \tau_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} l_{71} & l_{72} \\ l_{81} & l_{82} \\ l_{91} & l_{92} \end{bmatrix} \begin{Bmatrix} u_\alpha \\ u_\beta \end{Bmatrix} + \begin{bmatrix} l_{73} \\ l_{83} \\ l_{93} \end{bmatrix} u_\zeta + \begin{bmatrix} \tilde{c}_{13} \\ \tilde{c}_{23} \\ \tilde{c}_{36} \end{bmatrix} \sigma_\zeta, \tag{10}$$

where

$$l_{71} = \frac{Q_{11}}{\gamma_\alpha}\partial_\alpha + \frac{Q_{16}}{\gamma_\beta}\partial_\beta, \quad l_{72} = \frac{Q_{16}}{\gamma_\alpha}\partial_\alpha + \frac{Q_{12}}{\gamma_\beta}\partial_\beta, \quad l_{81} = \frac{Q_{12}}{\gamma_\alpha}\partial_\alpha + \frac{Q_{26}}{\gamma_\beta}\partial_\beta,$$

$$l_{82} = \frac{Q_{26}}{\gamma_\alpha}\partial_\alpha + \frac{Q_{22}}{\gamma_\beta}\partial_\beta, \quad l_{91} = \frac{Q_{16}}{\gamma_\alpha}\partial_\alpha + \frac{Q_{66}}{\gamma_\beta}\partial_\beta, \quad l_{92} = \frac{Q_{66}}{\gamma_\alpha}\partial_\alpha + \frac{Q_{26}}{\gamma_\beta}\partial_\beta,$$

$$l_{73} = \frac{Q_{11}}{\gamma_\alpha R_\alpha} + \frac{Q_{12}}{\gamma_\beta R_\beta}, \quad l_{83} = \frac{Q_{12}}{\gamma_\alpha R_\alpha} + \frac{Q_{22}}{\gamma_\beta R_\beta}, \quad l_{93} = \frac{Q_{16}}{\gamma_\alpha R_\alpha} + \frac{Q_{26}}{\gamma_\beta R_\beta}.$$

Let us consider a set of particular type of boundary conditions specified as follows: On the lateral surface the transverse load $q(\alpha, \beta, t)$ is prescribed,

$$[\tau_{\alpha\zeta} \;\; \tau_{\beta\zeta}] = [0 \;\; 0] \quad \text{on } \zeta = \pm h, \tag{11a}$$

$$\sigma_\zeta = q(\alpha, \beta, t) \quad \text{on } \zeta = h, \tag{11b}$$

$$\sigma_\zeta = 0 \quad \text{on } \zeta = -h. \tag{11c}$$

The edge boundary conditions require that one member of each pair of the following quantities be satisfied,

$$n_1\sigma_\alpha + n_2\tau_{\alpha\beta} = \bar{p}_1, \quad \text{or} \quad u_\alpha = \bar{u}_\alpha; \tag{12a}$$

$$n_1\tau_{\alpha\beta} + n_2\sigma_\beta = \bar{p}_2, \quad \text{or} \quad u_\beta = \bar{u}_\beta; \tag{12b}$$

$$n_1\tau_{\alpha\zeta} + n_2\tau_{\beta\zeta} = \bar{p}_3, \quad \text{or} \quad u_\zeta = \bar{u}_\zeta; \tag{12c}$$

where $\bar{p}_i$ ($i = 1, 2, 3$) are applied edge loads; $\bar{u}_\alpha$, $\bar{u}_\beta$ and $\bar{u}_\zeta$ are the prescribed edge displacements; n_1 and n_2 denote the outward normals at a point along the edge.

3. NONDIMENSIONALIZATION AND MULTIPLE SCALES

A key step in applying asymptotic expansion is to bring out in the formulation a small perturbation parameter characteristic to the problem. This is accomplished by making the basic equations dimensionless. To this end, let us define the dimensionless field variables as follows:

$$x = \frac{\alpha}{\sqrt{Rh}}, \quad y = \frac{\beta}{\sqrt{Rh}}, \quad z = \frac{\zeta}{h};$$

$$u = \frac{u_\alpha}{\sqrt{Rh}}, \quad v = \frac{u_\beta}{\sqrt{Rh}}, \quad w = \frac{u_\zeta}{R};$$

$$R_x = \frac{R_\alpha}{R} \quad R_y = \frac{R_\beta}{R};$$

$$\sigma_x = \frac{\sigma_\alpha}{Q}, \quad \sigma_y = \frac{\sigma_\beta}{Q}, \quad \tau_{xy} = \frac{\tau_{\alpha\beta}}{Q};$$

$$\tau_{xz} = \frac{\tau_{\alpha\zeta}}{Q\varepsilon}, \quad \tau_{yz} = \frac{\tau_{\beta\zeta}}{Q\varepsilon}, \quad \sigma_z = \frac{\sigma_\zeta}{Q\varepsilon^2}. \tag{13}$$

in which $\varepsilon^2 = h/R$ is a small parameter, usually much less than 1. R denotes a characteristic length of the shell. In the present analysis R is defined as a^2/h, where a is the smaller value of a_α and a_β. Q stands for a reference elastic modulus.

To provide flexibility in eliminating the secular terms in the asymptotic solution, it is constructive to introduce in the formulation the dimensionless multiple time scales defined by

$$\tau_k = \frac{\varepsilon^{2k}}{R}\sqrt{\frac{Q}{\rho_0}}\,t, \quad (k = 0, 1, 2, \ldots). \tag{14}$$

where ρ_0 represents a reference mass density.

Upon substituting (13)–(14) in (6)–(10), we obtain the dimensionless equations as follows:

$$w_{,z} = -\varepsilon^2[\tilde{l}_{31} \;\; \tilde{l}_{32}]\begin{Bmatrix} u \\ v \end{Bmatrix} - \varepsilon^2\tilde{l}_{33}w + \varepsilon^4\tilde{l}_{34}\sigma_z, \tag{15}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix}_{,z} = -\begin{bmatrix} \tilde{l}_{13} \\ \tilde{l}_{23} \end{bmatrix} w + \varepsilon^2\begin{bmatrix} \tilde{l}_{11} & 0 \\ 0 & \tilde{l}_{22} \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix} + \varepsilon^2\begin{bmatrix} \tilde{l}_{14} & \tilde{l}_{15} \\ \tilde{l}_{15} & \tilde{l}_{25} \end{bmatrix}\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} + \varepsilon^4\begin{bmatrix} \tilde{l}_{16} & \tilde{l}_{17} \\ \tilde{l}_{26} & \tilde{l}_{27} \end{bmatrix}\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}, \tag{16}$$

$$\begin{aligned}\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}_{,z} = &-\begin{bmatrix} \tilde{l}_{41} & \tilde{l}_{42} \\ \tilde{l}_{42} & \tilde{l}_{52} \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix} - \begin{bmatrix} \tilde{l}_{43} \\ \tilde{l}_{53} \end{bmatrix} w - \varepsilon^2\begin{bmatrix} \tilde{l}_{44} & 0 \\ 0 & \tilde{l}_{55} \end{bmatrix}\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} - \varepsilon^2\begin{bmatrix} \tilde{l}_{46} \\ \tilde{l}_{56} \end{bmatrix}\sigma_z - \varepsilon^4\begin{bmatrix} \tilde{l}_{45} & 0 \\ 0 & \tilde{l}_{45} \end{bmatrix}\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \\ &+ \rho_1\left[\frac{\partial^2}{\partial\tau_0^2} + 2\varepsilon^2\frac{\partial^2}{\partial\tau_0\,\partial\tau_1} + \varepsilon^4\left(2\frac{\partial^2}{\partial\tau_0\,\partial\tau_2} + \frac{\partial^2}{\partial\tau_1^2}\right) + \ldots\right]\begin{Bmatrix} u \\ v \end{Bmatrix},\end{aligned} \tag{17}$$

$$\begin{aligned}\sigma_{z,z} = {} & [\tilde{l}_{43} \;\; \tilde{l}_{53}]\begin{Bmatrix} u \\ v \end{Bmatrix} + \tilde{l}_{63}w - [\tilde{l}_{13} \;\; \tilde{l}_{23}]\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} - \varepsilon^2[\tilde{l}_{61} \;\; \tilde{l}_{62}]\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} - \varepsilon^2\tilde{l}_{64}\sigma_z - \varepsilon^4\tilde{l}_{65}\sigma_z \\ &+ \rho_2\left[\frac{\partial^2}{\partial\tau_0^2} + 2\varepsilon^2\frac{\partial^2}{\partial\tau_0\,\partial\tau_1} + \varepsilon^4\left(2\frac{\partial^2}{\partial\tau_0\,\partial\tau_2} + \frac{\partial^2}{\partial\tau_1^2}\right) + \ldots\right]w,\end{aligned} \tag{18}$$

where

$$\tilde{l}_{31} = \frac{\tilde{c}_{13}}{\gamma_\alpha}\partial_x + \frac{\tilde{c}_{36}}{\gamma_\beta}\partial_y, \quad \tilde{l}_{32} = \frac{\tilde{c}_{36}}{\gamma_\alpha}\partial_x + \frac{\tilde{c}_{23}}{\gamma_\beta}\partial_y, \quad \tilde{l}_{33} = \frac{\tilde{c}_{13}}{\gamma_\alpha R_x} + \frac{\tilde{c}_{23}}{\gamma_\beta R_y},$$

$$\tilde{l}_{34} = l_{34}Q, \quad \tilde{l}_{13} = \partial_x, \quad \tilde{l}_{23} = \partial_y, \quad \tilde{l}_{11} = \frac{1}{R_x} - \frac{z}{R_x}\partial_z, \quad \tilde{l}_{22} = \frac{1}{R_y} - \frac{z}{R_y}\partial_z,$$

$$\tilde{l}_{14} = l_{14}Q, \quad \tilde{l}_{15} = l_{15}Q, \quad \tilde{l}_{25} = l_{25}Q, \quad \tilde{l}_{16} = \frac{z}{R_x}\tilde{l}_{14}, \quad \tilde{l}_{17} = \frac{z}{R_x}\tilde{l}_{15},$$

$$\tilde{l}_{26} = \frac{z}{R_y}\tilde{l}_{15}, \quad \tilde{l}_{27} = \frac{z}{R_y}\tilde{l}_{25}, \quad \tilde{l}_{41} = \frac{\tilde{Q}_{11}\gamma_\beta}{\gamma_\alpha}\partial_{xx} + 2\tilde{Q}_{16}\partial_{xy} + \frac{\tilde{Q}_{66}\gamma_\alpha}{\gamma_\beta}\partial_{yy},$$

$$\tilde{l}_{42} = \frac{\tilde{Q}_{16}\gamma_\beta}{\gamma_\alpha}\partial_{xx} + (\tilde{Q}_{12} + \tilde{Q}_{66})\partial_{xy} + \frac{\tilde{Q}_{26}\gamma_\alpha}{\gamma_\beta}\partial_{yy},$$

$$\tilde{l}_{52} = \frac{\tilde{Q}_{66}\gamma_\beta}{\gamma_\alpha}\partial_{xx} + 2\tilde{Q}_{26}\partial_{xy} + \frac{\tilde{Q}_{22}\gamma_\alpha}{\gamma_\beta}\partial_{yy},$$

$$\tilde{l}_{43} = \left(\frac{\tilde{Q}_{11}\gamma_\beta}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{12}}{R_y}\right)\partial_x + \left(\frac{\tilde{Q}_{16}}{R_x} + \frac{\tilde{Q}_{26}\gamma_\alpha}{\gamma_\beta R_y}\right)\partial_y,$$

$$\tilde{l}_{53} = \left(\frac{\tilde{Q}_{16}\gamma_\beta}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{26}}{R_y}\right)\partial_x + \left(\frac{\tilde{Q}_{12}}{R_x} + \frac{\tilde{Q}_{22}\gamma_\alpha}{\gamma_\beta R_y}\right)\partial_y,$$

$$\tilde{l}_{44} = \frac{2}{R_x} + \frac{1}{R_y} + \left(\frac{z}{R_x} + \frac{z}{R_y}\right)\partial_z, \quad \tilde{l}_{55} = \frac{1}{R_x} + \frac{2}{R_y} + \left(\frac{z}{R_x} + \frac{z}{R_y}\right)\partial_z,$$

$$\tilde{l}_{45} = \frac{3z}{R_xR_y} + \frac{z^2}{R_xR_y}\frac{\partial}{\partial z}, \quad \tilde{l}_{46} = \tilde{c}_{13}\gamma_\beta\partial_x + \tilde{c}_{36}\gamma_\alpha\partial_y, \quad \tilde{l}_{56} = \tilde{c}_{36}\gamma_\beta\partial_x + \tilde{c}_{23}\gamma_\alpha\partial_y,$$

$$\tilde{l}_{61} = \frac{z}{R_y}\tilde{l}_{13}, \quad \tilde{l}_{62} = \frac{z}{R_x}\tilde{l}_{23}, \quad \tilde{l}_{63} = \frac{\tilde{Q}_{11}\gamma_\beta}{\gamma_\alpha R_x^2} + \frac{2\tilde{Q}_{12}}{R_xR_y} + \frac{\tilde{Q}_{22}\gamma_\alpha}{\gamma_\beta R_y^2},$$

$$\tilde{l}_{64} = \frac{1}{R_x} + \frac{1}{R_y} - \frac{\tilde{c}_{13}}{R_x} - \frac{\tilde{c}_{23}}{R_y} + \left(\frac{z}{R_x} + \frac{z}{R_y}\right)\partial_z,$$

$$\tilde{l}_{65} = (2-\tilde{c}_{13}-\tilde{c}_{23})\frac{z}{R_xR_y} + \frac{z^2}{R_xR_y}\partial_z, \quad \tilde{Q}_{ij} = \frac{Q_{ij}}{Q},$$

$$\rho_1 = \frac{h\rho}{R\rho_0}, \quad \rho_2 = \left(1+\frac{hz}{RR_x}\right)\left(1+\frac{hz}{RR_y}\right)\frac{\rho}{\rho_0}.$$

The dimensionless membrane stresses are given by

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \begin{bmatrix}\tilde{l}_{71} & \tilde{l}_{72}\\ \tilde{l}_{81} & \tilde{l}_{82}\\ \tilde{l}_{91} & \tilde{l}_{92}\end{bmatrix}\begin{Bmatrix}u\\ v\end{Bmatrix} + \begin{bmatrix}\tilde{l}_{73}\\ \tilde{l}_{83}\\ \tilde{l}_{93}\end{bmatrix}w + \varepsilon^2\begin{bmatrix}\tilde{c}_{13}\\ \tilde{c}_{23}\\ \tilde{c}_{36}\end{bmatrix}\sigma_z, \tag{19}$$

where

$$\tilde{l}_{71} = \frac{\tilde{Q}_{11}}{\gamma_\alpha}\partial_x + \frac{\tilde{Q}_{16}}{\gamma_\beta}\partial_y, \quad \tilde{l}_{72} = \frac{\tilde{Q}_{16}}{\gamma_\alpha}\partial_x + \frac{\tilde{Q}_{12}}{\gamma_\beta}\partial_y, \quad \tilde{l}_{81} = \frac{\tilde{Q}_{12}}{\gamma_\alpha}\partial_x + \frac{\tilde{Q}_{26}}{\gamma_\beta}\partial_y,$$

$$\tilde{l}_{82} = \frac{\tilde{Q}_{26}}{\gamma_\alpha}\partial_x + \frac{\tilde{Q}_{22}}{\gamma_\beta}\partial_y, \quad \tilde{l}_{91} = \frac{\tilde{Q}_{16}}{\gamma_\alpha}\partial_x + \frac{\tilde{Q}_{66}}{\gamma_\beta}\partial_y, \quad \tilde{l}_{92} = \frac{\tilde{Q}_{66}}{\gamma_\alpha}\partial_x + \frac{\tilde{Q}_{26}}{\gamma_\beta}\partial_y,$$

$$\tilde{l}_{73} = \frac{\tilde{Q}_{11}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{12}}{\gamma_\beta R_y}, \quad \tilde{l}_{83} = \frac{\tilde{Q}_{12}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{22}}{\gamma_\beta R_y}, \quad \tilde{l}_{93} = \frac{\tilde{Q}_{16}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{26}}{\gamma_\beta R_y}.$$

In view of (15)–(19) containing only even power terms of ε, we now expand the displacements and stresses in the form given by

$$f(x,y,z,\varepsilon) = f_{(0)}(x,y,z) + \varepsilon^2 f_{(1)}(x,y,z) + \varepsilon^4 f_{(2)}(x,y,z) + \dots \tag{20}$$

Substituting (20) into (15)–(19) and collecting coefficients of equal powers of ε, we obtain the following sets of equations at various levels.

Order ε^0:

$$w_{(0),z} = 0, \tag{21}$$

$$\mathbf{u}_{(0),z} = -\mathbf{L}_2 w_{(0)}, \tag{22}$$

$$\boldsymbol{\sigma}_{s(0),z} = -\mathbf{L}_6\mathbf{u}_{(0)} - \mathbf{L}_7 w_{(0)} + \rho_1\frac{\partial^2}{\partial\tau_0^2}\mathbf{u}_{(0)}, \tag{23}$$

$$\sigma_{z(0),z} = \mathbf{L}_{11}\mathbf{u}_{(0)} + \tilde{l}_{63}w_{(0)} - \mathbf{L}_{12}\boldsymbol{\sigma}_{s(0)} + \rho_2\frac{\partial^2}{\partial\tau_0^2}w_{(0)}, \tag{24}$$

$$\boldsymbol{\sigma}_{m(0)} = \mathbf{L}_{14}\mathbf{u}_{(0)} + \mathbf{L}_{15}w_{(0)}, \tag{25}$$

Order ε^2:

$$w_{(1),z} = -\mathbf{L}_1\mathbf{u}_{(0)} - \tilde{l}_{33}w_{(0)}, \tag{26}$$

$$\mathbf{u}_{(1),z} = \mathbf{L}_3\mathbf{u}_{(0)} - \mathbf{L}_2 w_{(1)} + \mathbf{L}_4\boldsymbol{\sigma}_{s(0)}, \tag{27}$$

$$\boldsymbol{\sigma}_{s(1),z} = -\mathbf{L}_6\mathbf{u}_{(1)} - \mathbf{L}_7 w_{(1)} - \mathbf{L}_8\boldsymbol{\sigma}_{s(0)} - \mathbf{L}_9\boldsymbol{\sigma}_{z(0)} + \rho_1\frac{\partial^2}{\partial\tau_0^2}\mathbf{u}_{(1)} + 2\rho_1\frac{\partial^2}{\partial\tau_0\,\partial\tau_1}\mathbf{u}_{(0)}, \tag{28}$$

$$\sigma_{z(1),z} = \mathbf{L}_{11}\mathbf{u}_{(1)} + \tilde{l}_{63}w_{(1)} - \mathbf{L}_{12}\boldsymbol{\sigma}_{s(1)} - \mathbf{L}_{13}\boldsymbol{\sigma}_{s(0)} - \tilde{l}_{64}\sigma_{z(0)} + \rho_2\frac{\partial^2}{\partial\tau_0^2}w_{(1)} + 2\rho_2\frac{\partial^2}{\partial\tau_0\,\partial\tau_1}w_{(0)}, \tag{29}$$

$$\boldsymbol{\sigma}_{m(1)} = \mathbf{L}_{14}\mathbf{u}_{(1)} + \mathbf{L}_{15}w_{(1)} + \mathbf{L}_{16}\sigma_{z(0)}, \tag{30}$$

Order ε^{2k} $(k = 2, 3, \ldots)$:

$$w_{(k),z} = -\mathbf{L}_1\mathbf{u}_{(k-1)} - \tilde{l}_{33}w_{(k-1)} + \tilde{l}_{34}\sigma_{z(k-2)}, \tag{31}$$

$$\mathbf{u}_{(k),z} = \mathbf{L}_3\mathbf{u}_{(k-1)} - \mathbf{L}_2 w_{(k)} + \mathbf{L}_4\boldsymbol{\sigma}_{s(k-1)} + \mathbf{L}_5\boldsymbol{\sigma}_{s(k-2)}, \tag{32}$$

$$\begin{aligned}\boldsymbol{\sigma}_{s(k),z} = {} & -\mathbf{L}_6\mathbf{u}_{(k)} - \mathbf{L}_7 w_{(k)} - \mathbf{L}_8\boldsymbol{\sigma}_{s(k-1)} - \mathbf{L}_9\sigma_{z(k-1)} - \mathbf{L}_{10}\boldsymbol{\sigma}_{s(k-2)} \\ & + \left[\rho_1\frac{\partial^2}{\partial\tau_0^2}\mathbf{u}_{(k)} + 2\rho_1\frac{\partial^2}{\partial\tau_0\,\partial\tau_1}\mathbf{u}_{(k-1)} \right. \\ & \left. + \cdots + \rho_1\left(\frac{\partial^2}{\partial\tau_0\,\partial\tau_k} + \frac{\partial^2}{\partial\tau_1\,\partial\tau_{k-1}} + \cdots + \frac{\partial^2}{\partial\tau_k\,\partial\tau_0}\right)\mathbf{u}_{(0)}\right],\end{aligned} \tag{33}$$

$$\begin{aligned}\sigma_{z(k),z} = {} & \mathbf{L}_{11}\mathbf{u}_{(k)} + \tilde{l}_{63}w_{(k)} - \mathbf{L}_{12}\boldsymbol{\sigma}_{s(k)} - \mathbf{L}_{13}\boldsymbol{\sigma}_{s(k-1)} - \tilde{l}_{64}\sigma_{z(k-1)} \\ & - \tilde{l}_{65}\sigma_{z(k-2)} + \left[\rho_2\frac{\partial^2}{\partial\tau_0^2}w_{(k)} + 2\rho_2\frac{\partial^2}{\partial\tau_0\,\partial\tau_k}w_{(k-1)} \right. \\ & \left. + \cdots + \rho_2\left(\frac{\partial^2}{\partial\tau_0\,\partial\tau_k} + \frac{\partial^2}{\partial\tau_1\,\partial\tau_{k-1}} + \cdots + \frac{\partial^2}{\partial\tau_k\,\partial\tau_0}\right)w_{(0)}\right],\end{aligned} \tag{34}$$

$$\boldsymbol{\sigma}_{m(k)} = \mathbf{L}_{14}\mathbf{u}_{(k)} + \mathbf{L}_{15}w_{(k)} + \mathbf{L}_{16}\sigma_{z(k-1)}, \tag{35}$$

where

$$\mathbf{u} = \begin{Bmatrix} u \\ v \end{Bmatrix}, \quad \boldsymbol{\sigma}_s = \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}, \quad \boldsymbol{\sigma}_m = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}, \quad \mathbf{L}_1 = [\tilde{l}_{31}\ \ \tilde{l}_{32}],$$

$$\mathbf{L}_2 = \begin{bmatrix} \tilde{l}_{13} \\ \tilde{l}_{23} \end{bmatrix}, \quad \mathbf{L}_3 = \begin{bmatrix} \tilde{l}_{11} & 0 \\ 0 & \tilde{l}_{22} \end{bmatrix}, \quad \mathbf{L}_4 = \begin{bmatrix} \tilde{l}_{14} & \tilde{l}_{15} \\ \tilde{l}_{15} & \tilde{l}_{25} \end{bmatrix}, \quad \mathbf{L}_5 = \begin{bmatrix} \tilde{l}_{16} & \tilde{l}_{17} \\ \tilde{l}_{26} & \tilde{l}_{27} \end{bmatrix},$$

$$\mathbf{L}_6 = \begin{bmatrix} \tilde{l}_{41} & \tilde{l}_{42} \\ \tilde{l}_{42} & \tilde{l}_{52} \end{bmatrix}, \quad \mathbf{L}_7 = \begin{bmatrix} \tilde{l}_{43} \\ \tilde{l}_{53} \end{bmatrix}, \quad \mathbf{L}_8 = \begin{bmatrix} \tilde{l}_{44} & 0 \\ 0 & \tilde{l}_{55} \end{bmatrix}, \quad \mathbf{L}_9 = \begin{bmatrix} \tilde{l}_{46} \\ \tilde{l}_{56} \end{bmatrix}, \quad \mathbf{L}_{10} = \begin{bmatrix} \tilde{l}_{45} & 0 \\ 0 & \tilde{l}_{45} \end{bmatrix},$$

$$\mathbf{L}_{11} = [\tilde{l}_{43}\ \ \tilde{l}_{53}], \quad \mathbf{L}_{12} = [\tilde{l}_{13}\ \ \tilde{l}_{23}], \quad \mathbf{L}_{13} = [\tilde{l}_{61}\ \ \tilde{l}_{62}],$$

$$\mathbf{L}_{14}=\begin{bmatrix}\tilde{l}_{71} & \tilde{l}_{72}\\ \tilde{l}_{81} & \tilde{l}_{82}\\ \tilde{l}_{91} & \tilde{l}_{92}\end{bmatrix},\quad \mathbf{L}_{15}=\begin{bmatrix}\tilde{l}_{73}\\ \tilde{l}_{83}\\ \tilde{l}_{93}\end{bmatrix},\quad \mathbf{L}_{16}=\begin{bmatrix}\tilde{c}_{13}\\ \tilde{c}_{23}\\ \tilde{c}_{36}\end{bmatrix}.$$

The associated dimensionless boundary conditions are

Order ε^0:

$$[\tau_{xz(0)}\ \ \tau_{yz(0)}]=[0\ \ 0]\quad \text{on}\quad z=\pm 1, \tag{36}$$

$$\sigma_{z(0)}=\tilde{q}(x,y,t)\quad \text{on}\quad z=1, \tag{37}$$

$$\sigma_{z(0)}=0\quad \text{on}\quad z=-1. \tag{38}$$

Along the edges one member of each pair of the following quantities must be satisfied:

$$n_1\sigma_{x(0)}+n_2\tau_{xy(0)}=\tilde{p}_1,\quad \text{or}\quad u_{(0)}=\tilde{u}, \tag{39}$$

$$n_1\tau_{xy(0)}+n_2\sigma_{y(0)}=\tilde{p}_2,\quad \text{or}\quad v_{(0)}=\tilde{v}, \tag{40}$$

$$n_1\sigma_{xz(0)}+n_2\tau_{yz(0)}=\tilde{p}_3,\quad \text{or}\quad w_{(0)}=\tilde{w}. \tag{41}$$

Order ε^{2k} $(k=1,2,3,\ldots)$:

$$[\tau_{xz(k)}\ \tau_{yz(k)}]=[0\ \ 0]\quad \text{on}\quad z=\pm 1, \tag{42}$$

$$\sigma_{z(k)}=0\quad \text{on}\quad z=\pm 1. \tag{43}$$

Along the edges one member of each pair of the following quantities must be satisfied:

$$n_1\sigma_{x(k)}+n_2\tau_{xy(k)}=0,\quad \text{or}\quad u_{(k)}=0, \tag{44}$$

$$n_1\tau_{xy(k)}+n_2\sigma_{y(k)}=0,\quad \text{or}\quad v_{(k)}=0, \tag{45}$$

$$n_1\sigma_{xz(k)}+n_2\tau_{yz(k)}=0,\quad \text{or}\quad w_{(k)}=0, \tag{46}$$

where $\tilde{q}=q/Q\varepsilon^2$, $\tilde{p}_k=\bar{p}_k/Q$, $\tilde{u}=\bar{u}_\alpha/\sqrt{Rh}$, $\tilde{v}=\bar{u}_\beta/\sqrt{Rh}$ and $\tilde{w}=\bar{u}_\zeta/R$.

4. SUCCESSIVE INTEGRATION AND CST

The asymptotic equations can be integrated with respect to z in succession. The associated lateral boundary conditions will be satisfied in the course of the integration. As a result, we obtain at the leading order

$$w_{(0)}=w_0(x,y,\tau_0,\tau_1,\ldots), \tag{47}$$

$$\mathbf{u}_{(0)}=\mathbf{u}_0(x,y,\tau_0,\tau_1,\ldots)-z\mathbf{D}^{\mathrm{T}}w_0, \tag{48}$$

$$\boldsymbol{\sigma}_{s(0)}=-\int_{-1}^{z}\{\mathbf{L}_6(\mathbf{u}_0-\eta\mathbf{D}^{\mathrm{T}}w_0)+\mathbf{L}_7w_0\}\,\mathrm{d}\eta+\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}\rho_1(\mathbf{u}_0-\eta\mathbf{D}^{\mathrm{T}}w_0)\,\mathrm{d}\eta, \tag{49}$$

$$\sigma_{z(0)} = \int_{-1}^{z} [\mathbf{L}_{11}(\mathbf{u}_0-\eta\mathbf{D}^{\mathrm{T}}w_0)+\tilde{l}_{63}w_0]\,\mathrm{d}\eta+\int_{-1}^{z}(z-\eta)\mathbf{D}[\mathbf{L}_6(\mathbf{u}_0-\eta\mathbf{D}^{\mathrm{T}}w_0)+\mathbf{L}_7w_0]\,\mathrm{d}\eta$$
$$+\left(\int_{-1}^{z}\rho_2\,\mathrm{d}\eta\right)\frac{\partial^2 w_0}{\partial\tau_0^2}-\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}\rho_1(z-\eta)\mathbf{D}(\mathbf{u}_0-\eta\mathbf{D}^{\mathrm{T}}w_0)\,\mathrm{d}\eta. \quad (50)$$

where $w_0(x,y,\tau_0,\tau_1,\ldots)$, $\mathbf{u}_0=[u_0(x,y,\tau_0,\tau_1,\ldots)\;\; v_0(x,y,\tau_0,\tau_1,\ldots)]^{\mathrm{T}}$ represent the displacements on the middle surface and $\mathbf{D}=[\partial_x\;\partial_y]$. The displacements in (47) and (48) are of the Kirchhoff-Love's type in the CST.

Consideration of the boundary conditions follows. The boundary conditions on $z=-1$ are satisfied by (49) and (50). After imposing the remaining lateral boundary conditions (36)–(37) on (49)–(50), we obtain

$$\int_{-1}^{1}\{\mathbf{L}_6(\mathbf{u}_0-\eta\mathbf{D}^{\mathrm{T}}w_0)+\mathbf{L}_7w_0\}\,\mathrm{d}\eta=\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{1}\rho_1(\mathbf{u}_0-\eta\mathbf{D}^{\mathrm{T}}w_0)\,\mathrm{d}\eta, \quad (51)$$

$$\int_{-1}^{1}[\mathbf{L}_{11}(\mathbf{u}_0-\eta\mathbf{D}^{\mathrm{T}}w_0)+\tilde{l}_{63}w_0]\,\mathrm{d}\eta+\int_{-1}^{1}(1-\eta)\mathbf{D}[\mathbf{L}_6(\mathbf{u}_0-\eta\mathbf{D}^{\mathrm{T}}w_0)+\mathbf{L}_7w_0]\,\mathrm{d}\eta$$
$$=\tilde{q}-\left(\int_{-1}^{1}\rho_2\,\mathrm{d}\eta\right)\frac{\partial^2 w_0}{\partial\tau_0^2}+\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{1}\rho_1(1-\eta)\mathbf{D}(\mathbf{u}_0-\eta\mathbf{D}^{\mathrm{T}}w_0)\,\mathrm{d}\eta. \quad (52)$$

After a simple manipulation, (51) and (52) can be written explicitly as

$$K_{11}u_0+K_{12}v_0+K_{13}w_0=I_{10}\frac{\partial^2 u_0}{\partial\tau_0^2}-I_{11}\frac{\partial^2}{\partial\tau_0^2}(w_{0,x}), \quad (53)$$

$$K_{21}u_0+K_{22}v_0+K_{23}w_0=I_{10}\frac{\partial^2 v_0}{\partial\tau_0^2}-I_{11}\frac{\partial^2}{\partial\tau_0^2}(w_{0,y}), \quad (54)$$

$$K_{31}u_0+K_{32}v_0+K_{33}w_0=\tilde{q}-I_{20}\frac{\partial^2 w_0}{\partial\tau_0^2}+I_{12}\frac{\partial^2}{\partial\tau_0^2}(w_{0,xx}+w_{0,yy})-I_{11}\frac{\partial^2}{\partial\tau_0^2}(u_{0,x}+v_{0,y}), \quad (55)$$

in which

$$K_{11}=\hat{A}_{11}\partial_{xx}+2\tilde{A}_{16}\partial_{xy}+\bar{A}_{66}\partial_{yy},$$
$$K_{12}=K_{21}=\hat{A}_{16}\partial_{xx}+(\tilde{A}_{12}+\tilde{A}_{66})\partial_{xy}+\bar{A}_{26}\partial_{yy},$$
$$K_{13}=K_{31}=-\hat{B}_{11}\partial_{xxx}-(2\tilde{B}_{16}+\hat{B}_{16})\partial_{xxy}-(\bar{B}_{66}+\bar{B}_{12}+\tilde{B}_{66})\partial_{xyy}$$
$$-\bar{B}_{26}\partial_{yyy}+\left(\frac{\hat{A}_{11}}{R_x}+\frac{\tilde{A}_{12}}{R_y}\right)\partial_x+\left(\frac{\tilde{A}_{16}}{R_x}+\frac{\bar{A}_{26}}{R_y}\right)\partial_y,$$
$$K_{22}=\hat{A}_{66}\partial_{xx}+2\tilde{A}_{26}\partial_{xy}+\bar{A}_{22}\partial_{yy},$$
$$K_{23}=K_{32}=-\hat{B}_{16}\partial_{xxx}-(\hat{B}_{66}+\tilde{B}_{12}+\tilde{B}_{66})\partial_{xxy}-(\bar{B}_{26}+2\tilde{B}_{26})\partial_{xyy}$$
$$-\bar{B}_{22}\partial_{yyy}+\left(\frac{\hat{A}_{16}}{R_x}+\frac{\tilde{A}_{26}}{R_y}\right)\partial_x+\left(\frac{\tilde{A}_{12}}{R_x}+\frac{\bar{A}_{22}}{R_y}\right)\partial_y,$$

$$K_{33} = \hat{D}_{11}\,\partial_{xxxx} + (2\tilde{D}_{16}+2\hat{D}_{16})\,\partial_{xxxy} + (2\tilde{D}_{12}+\bar{D}_{66}+2\tilde{D}_{66}+\hat{D}_{66})\,\partial_{xxyy} + (2\bar{D}_{26}+2\tilde{D}_{26})\,\partial_{xyyy} + \bar{D}_{22}\,\partial_{yyyy} - 2\left(\frac{\hat{B}_{11}}{R_x}+\frac{\tilde{B}_{12}}{R_y}\right)\partial_{xx} - 2\left(\frac{\tilde{B}_{12}}{R_x}+\frac{\bar{B}_{22}}{R_y}\right)\partial_{yy} - 2\left(\frac{\tilde{B}_{16}}{R_x}+\frac{\hat{B}_{16}}{R_x}+\frac{\bar{B}_{26}}{R_y}+\frac{\tilde{B}_{26}}{R_y}\right)\partial_{xy} + \left(\frac{\hat{A}_{11}}{R_x^2}+\frac{2\tilde{A}_{12}}{R_xR_y}+\frac{\bar{A}_{22}}{R_y^2}\right),$$

$$\hat{A}_{ij} = \int_{-1}^{1}\frac{\tilde{Q}_{ij}\gamma_\beta}{\gamma_\alpha}\,dz,\quad \bar{A}_{ij} = \int_{-1}^{1}\frac{\tilde{Q}_{ij}\gamma_\alpha}{\gamma_\beta}\,dz,\quad \tilde{A}_{ij} = \int_{-1}^{1}\tilde{Q}_{ij}\,dz,$$

$$\hat{B}_{ij} = \int_{-1}^{1}\frac{\tilde{Q}_{ij}\gamma_\beta}{\gamma_\alpha}z\,dz,\quad \bar{B}_{ij} = \int_{-1}^{1}\frac{\tilde{Q}_{ij}\gamma_\alpha}{\gamma_\beta}z\,dz,\quad \tilde{B}_{ij} = \int_{-1}^{1}\tilde{Q}_{ij}z\,dz,$$

$$\hat{D}_{ij} = \int_{-1}^{1}\frac{\tilde{Q}_{ij}\gamma_\beta}{\gamma_\alpha}z^2\,dz,\quad \bar{D}_{ij} = \int_{-1}^{1}\frac{\tilde{Q}_{ij}\gamma_\alpha}{\gamma_\beta}z^2\,dz,\quad \tilde{D}_{ij} = \int_{-1}^{1}\tilde{Q}_{ij}z^2\,dz,$$

$$I_{10} = \int_{-1}^{1}\rho_1\,dz,\quad I_{11} = \int_{-1}^{1}\rho_1 z\,dz,\quad I_{12} = \int_{-1}^{1}\rho_1 z^2\,dz,\quad I_{20} = \int_{-1}^{1}\rho_2\,dz.$$

Examining the governing equations for displacements in CST (Leissa and Qatu, 1991), we find that the CST equations are reproduced from (53)–(55) after imposing an assumption of the thin shell: $z/R_\alpha \ll 1$ and $z/R_\beta \ll 1$. In the present notation, it implies $\gamma_\alpha \approx 1$, $\gamma_\beta \approx 1$, $\bar{A}_{ij} = \hat{A}_{ij} = \tilde{A}_{ij} = A_{ij}/Qh$, $\bar{B}_{ij} = \hat{B}_{ij} = \tilde{B}_{ij} = B_{ij}/Qh^2$, $\bar{D}_{ij} = \hat{D}_{ij} = \tilde{D}_{ij} = D_{ij}/Qh^3$ in which A_{ij}, B_{ij}, D_{ij} are the so called extension, extension-bending and bending stiffnesses, respectively. The CST thus has been shown to be a first-order approximation to the three-dimensional theory. Solutions of (53)–(55) must be supplemented with the edge boundary conditions (39)–(41) to constitute a well-posed problem. Once u_0, v_0 and w_0 are determined, the leading-order displacements are given by (47)–(48), the transverse shear and normal stresses by (49)–(50), and the membrane stresses by (25).

Carrying on the analysis to order ε^2 by integrating (26)–(29) in succession, we readily obtain

$$w_{(1)} = w_1(x,y,\tau_0,\tau_1,\ldots) + \phi_{31}(x,y,z,\tau_0,\tau_1,\ldots), \tag{56}$$

$$\mathbf{u}_{(1)} = \mathbf{u}_1(x,y,\tau_0,\tau_1,\ldots) - z\mathbf{D}^{\mathrm{T}}w_1 + \boldsymbol{\phi}_1(x,y,z,\tau_0,\tau_1,\ldots), \tag{57}$$

$$\boldsymbol{\sigma}_{s(1)} = -\int_{-1}^{z}[\mathbf{L}_6(\mathbf{u}_1-\eta\mathbf{D}^{\mathrm{T}}w_1)+\mathbf{L}_7w_1]\,d\eta + \mathbf{f}_1(x,y,z,\tau_0,\tau_1,\ldots) + \frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}\rho_1(\mathbf{u}_1-\eta\mathbf{D}^{\mathrm{T}}w_1)\,d\eta, \tag{58}$$

$$\sigma_{z(1)} = \int_{-1}^{z}[\mathbf{L}_{11}(\mathbf{u}_1-\eta\mathbf{D}^{\mathrm{T}}w_1)+l_{63}w_1]\,d\eta + \int_{-1}^{z}(z-\eta)\mathbf{D}[\mathbf{L}_6(\mathbf{u}_1-\eta\mathbf{D}^{\mathrm{T}}w_1)+\mathbf{L}_7w_1]\,d\eta - f_{31}(x,y,z,\tau_0,\tau_1,\ldots) - \frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}(z-\eta)\mathbf{D}(\mathbf{u}_1-\eta\mathbf{D}^{\mathrm{T}}w_1)\,d\eta + \left(\int_{-1}^{z}\rho_2\,d\eta\right)\frac{\partial^2 w_1}{\partial\tau_0^2}, \tag{59}$$

where w_1 and $\mathbf{u}_1$ represent the modifications to the displacements on the middle surface, and

$$\mathbf{u}_1 = [u_1(x,y,\tau_0,\tau_1,\ldots)\ v_1(x,y,\tau_0,\tau_1,\ldots)]^{\mathrm{T}},$$

$$f_{31}(x,y,z,\tau_0,\tau_1,\ldots) = -\int_{-1}^{z}(\mathbf{L}_{11}\boldsymbol{\phi}_1 + \tilde{l}_{63}\phi_{31} - \mathbf{L}_{12}\mathbf{f}_1 - \mathbf{L}_{13}\boldsymbol{\sigma}_{s(0)} - \tilde{l}_{64}\sigma_{z(0)})\,d\eta$$

$$-\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}\rho_2\phi_{31}\,d\eta - 2\left(\int_{-1}^{z}\rho_2\,d\eta\right)\frac{\partial^2 w_0}{\partial\tau_0\,\partial\tau_1},$$

$$\mathbf{f}_1 = \begin{Bmatrix} f_{11}(x,y,z,\tau_0,\tau_1,\ldots) \\ f_{21}(x,y,z,\tau_0,\tau_1,\ldots)\end{Bmatrix} = -\int_{-1}^{z}(\mathbf{L}_6\boldsymbol{\phi}_1 + \mathbf{L}_7\phi_{31} + \mathbf{L}_8\boldsymbol{\sigma}_{s(0)} + \mathbf{L}_9\sigma_{z(0)})\,d\eta$$

$$+\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}\rho_1\boldsymbol{\phi}_1\,d\eta + \frac{\partial^2}{\partial\tau_0\,\partial\tau_1}\left(2\int_{-1}^{z}\rho_1\mathbf{u}_{(0)}\right)d\eta,$$

$$\phi_{31}(x,y,z,\tau_0,\tau_1,\ldots) = -\int_0^{z}(\mathbf{L}_1\mathbf{u}_{(0)} + \tilde{l}_{33}w_{(0)})\,d\eta,$$

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} \phi_{11}(x,y,z,\tau_0,\tau_1,\ldots) \\ \phi_{21}(x,y,z,\tau_0,\tau_1,\ldots)\end{Bmatrix} = \int_0^{z}(\mathbf{L}_3\mathbf{u}_{(0)} + \mathbf{L}_4\boldsymbol{\sigma}_{s(0)} - \mathbf{L}_2\boldsymbol{\phi}_1)\,d\eta.$$

Upon imposing the associated lateral boundary conditions (42)–(43) on (58) and (59), we obtain the CST type equations with nonhomogeneous terms carried over from the leading-order solution.

$$K_{11}u_1 + K_{12}v_1 + K_{13}w_1 = I_{10}\frac{\partial^2 u_1}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}(w_{1,x}) + f_{11}(x,y,1), \tag{60}$$

$$K_{21}u_1 + K_{22}v_1 + K_{23}w_1 = I_{10}\frac{\partial^2 v_1}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}(w_{1,y}) + f_{21}(x,y,1), \tag{61}$$

$$K_{31}u_1 + K_{32}v_1 + K_{33}w_1 = -I_{20}\frac{\partial^2 w_1}{\partial\tau_0^2} + I_{12}\frac{\partial^2}{\partial\tau_0^2}(w_{1,xx} + w_{1,yy}) - I_{11}\frac{\partial^2}{\partial\tau_0^2}(u_{1,x} + v_{1,y})$$
$$+ f_{31}(x,y,1) - \frac{\partial f_{11}(x,y,1)}{\partial x} - \frac{\partial f_{21}(x,y,1)}{\partial y}. \tag{62}$$

The governing equations at the ε^{2k} order are obtained by integrating (31)–(34) in succession. The resulting equations are

$$K_{11}u_k + K_{12}v_k + K_{13}w_k = I_{10}\frac{\partial^2 u_k}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}(w_{k,x}) + f_{1k}(x,y,1), \tag{63}$$

$$K_{21}u_k + K_{22}v_k + K_{23}w_k = I_{10}\frac{\partial^2 v_k}{\partial\tau_0^2} - I_{11}\frac{\partial^2}{\partial\tau_0^2}(w_{k,y}) + f_{2k}(x,y,1), \tag{64}$$

$$K_{31}u_k + K_{32}v_k + K_{33}w_k = -I_{20}\frac{\partial^2 w_k}{\partial\tau_0^2} + I_{12}\frac{\partial^2}{\partial\tau_0^2}(w_{k,xx} + w_{k,yy}) - I_{11}\frac{\partial^2}{\partial\tau_0^2}(u_{k,x} + v_{k,y})$$
$$+ f_{3k}(x,y,1) - \frac{\partial f_{1k}(x,y,1)}{\partial x} - \frac{\partial f_{2k}(x,y,1)}{\partial y}, \tag{65}$$

in which

$$f_{3k}(x,y,z,\tau_0,\tau_1,\ldots) = -\int_{-1}^{z} (\mathbf{L}_{11}\boldsymbol{\phi}_k + \tilde{l}_{63}\phi_{3k} - \mathbf{L}_{12}\mathbf{f}_k - \mathbf{L}_{13}\boldsymbol{\sigma}_{s(k-1)} - \tilde{l}_{64}\sigma_{z(k-1)} - \tilde{l}_{65}\sigma_{z(k-2)})\,\mathrm{d}\eta$$
$$-\left[\frac{\partial^2}{\partial\tau_0^2}\left(\int_{-1}^{z}\rho_2\phi_{3k}\,\mathrm{d}\eta\right) + \frac{\partial^2}{\partial\tau_0\,\partial\tau_k}\left(2\int_{-1}^{z}\rho_2 w_{(k-1)}\,\mathrm{d}\eta\right)\right.$$
$$\left. + \cdots + \left(\frac{\partial^2}{\partial\tau_0\,\partial\tau_k} + \frac{\partial^2}{\partial\tau_1\,\partial\tau_{k-1}} + \cdots + \frac{\partial^2}{\partial\tau_k\,\partial\tau_0}\right)\left(\int_{-1}^{z}\rho_2 w_0\,\mathrm{d}\eta\right)\right],$$

$$\mathbf{f}_k = \begin{Bmatrix} f_{1k}(x,y,z,\tau_0,\tau_1,\ldots) \\ f_{2k}(x,y,z,\tau_0,\tau_1,\ldots) \end{Bmatrix} = -\int_{-1}^{z} (\mathbf{L}_6\boldsymbol{\phi}_k + \mathbf{L}_7\phi_{3k} + \mathbf{L}_8\boldsymbol{\sigma}_{s(k-1)} + \mathbf{L}_{10}\boldsymbol{\sigma}_{s(k-2)} + \mathbf{L}_9\sigma_{z(k-1)})\,\mathrm{d}\eta$$
$$+\left[\frac{\partial^2}{\partial\tau_0^2}\int_{-1}^{z}\rho_1\boldsymbol{\phi}_k\,\mathrm{d}\eta + \frac{\partial^2}{\partial\tau_0\,\partial\tau_1}\left(2\int_{-1}^{z}\rho_1\mathbf{u}_{(k-1)}\,\mathrm{d}\eta\right)\right.$$
$$\left. + \cdots + \left(\frac{\partial^2}{\partial\tau_0\,\partial\tau_k} + \frac{\partial^2}{\partial\tau_1\,\partial\tau_{k-1}} + \cdots + \frac{\partial^2}{\partial\tau_k\,\partial\tau_0}\right)\left(\int_{-1}^{z}\rho_1\mathbf{u}_{(0)}\,\mathrm{d}\eta\right)\right],$$

$$\phi_{3k}(x,y,z,\tau_0,\tau_1,\ldots) = -\int_0^z (\mathbf{L}_1\mathbf{u}_{(k-1)} + \tilde{l}_{33}w_{(k-1)} - \tilde{l}_{34}\sigma_{z(k-2)})\,\mathrm{d}\eta,$$

$$\boldsymbol{\phi}_k = \begin{Bmatrix} \phi_{1k}(x,y,z,\tau_0,\tau_1,\ldots) \\ \phi_{2k}(x,y,z,\tau_0,\tau_1,\ldots) \end{Bmatrix} = \int_0^z (\mathbf{L}_3\mathbf{u}_{(k-1)} + \mathbf{L}_4\boldsymbol{\sigma}_{s(k-1)} + \mathbf{L}_5\boldsymbol{\sigma}_{s(k-2)} - \mathbf{L}_2\phi_{3k})\,\mathrm{d}\eta.$$

The accuracy of the asymptotic solution can be roughly estimated by examining the mathematical order. As the leading-order solution is obtained by truncating all the higher-order terms beyond ε^2-order, taking into account the orders in the nondimensionalization, we find that in the absence of transverse loads it gives the transverse normal stress σ_ζ accurate to $O(\varepsilon^4)$, the displacements u_α, u_β and transverse shear stresses $\tau_{\alpha\zeta}$, $\tau_{\beta\zeta}$ accurate to $O(\varepsilon^3)$ and the displacement u_ζ and membrane stresses σ_α, σ_β, $\tau_{\alpha\beta}$ accurate to $O(\varepsilon^2)$. The accuracy increases according to the power of ε^2 as we carry out the analysis one level higher.

5. SOLVABILITY CONDITIONS

It is well known in asymptotic analysis that the asymptotic solution will not be valid unless the expansion is uniform and free of secular terms (Nayfeh, 1981). As the governing equations at various levels are in the same form except for the nonhomogeneous terms involving lower-order solutions, it is necessary to investigate the solvability conditions under which the equations possess solutions that are bounded and free of secular terms.

Let us derive the solvability conditions for the ε^{2k}-order equations. To simplify the notation, we rewrite (63)–(65) in a differential operator form as follows:

$$L_1(u_k, v_k, w_k) = f_{1k}(x,y,1), \tag{66}$$

$$L_2(u_k, v_k, w_k) = f_{2k}(x,y,1), \tag{67}$$

$$L_3(u_k, v_k, w_k) = f_{3k}(x,y,1) - \frac{\partial f_{1k}(x,y,1)}{\partial x} - \frac{\partial f_{2k}(x,y,1)}{\partial y}. \tag{68}$$

Multiplying (66)–(68) respectively by the weighting functions φ_1, φ_2 and φ_3 (to be determined in the course of derivation), integrating over the domain of the middle surface and then we add them together in the following form

$$\int_A \varphi_1[L_1(u_k, v_k, w_k) - f_{1k}(x, y, 1)]\,dA + \int_A \varphi_2[L_2(u_k, v_k, w_k) - f_{2k}(x, y, 1)]\,dA$$
$$- \int_A \varphi_3\left[L_3(u_k, v_k, w_k) - f_{3k}(x, y, 1) + \frac{\partial f_{1k}(x, y, 1)}{\partial x} + \frac{\partial f_{2k}(x, y, 1)}{\partial y}\right] dA = 0. \quad (69)$$

The differential operation in the integrals of (69) can be transfered from u_k, v_k, w_k to φ_1, φ_2, φ_3 by applying Green's theorem. After a lengthy but straightforward manipulation, we arrive at

$$\int_A u_k L_1(\varphi_1, \varphi_2, \varphi_3)\,dA + \int_A v_k L_2(\varphi_1, \varphi_2, \varphi_3)\,dA - \int_A w_k L_3(\varphi_1, \varphi_2, \varphi_3)\,dA$$
$$+ \int_\Gamma \{u_k[L_4(\varphi_1, \varphi_2, \varphi_3)n_x + L_5(\varphi_1, \varphi_2, \varphi_3)n_y]$$
$$- \varphi_1[L_4(u_k, v_k, w_k)n_x + L_5(u_k, v_k, w_k)n_y]\,d\Gamma\}$$
$$+ \int_\Gamma \{v_k[L_6(\varphi_1, \varphi_2, \varphi_3)n_x + L_7(\varphi_1, \varphi_2, \varphi_3)n_y]$$
$$- \varphi_2[L_6(u_k, v_k, w_k)n_x + L_7(u_k, v_k, w_k)n_y]\,d\Gamma\}$$
$$+ \int_\Gamma \{w_k[L_8(\varphi_1, \varphi_2, \varphi_3)n_x + L_9(\varphi_1, \varphi_2, \varphi_3)n_y]$$
$$- \varphi_3[L_8(u_k, v_k, w_k)n_x + L_9(u_k, v_k, w_k)n_y]\,d\Gamma\}$$
$$- \int_A \left[\varphi_1 f_{1k} + \varphi_2 f_{2k} + \varphi_3\left(\frac{\partial f_{1k}}{\partial x} + \frac{\partial f_{2k}}{\partial y} - f_{3k}\right)\right] dA = 0, \quad (70)$$

where Γ denotes the boundary of the domain A, n_x and n_y are the components of an outward normal at on Γ.

If we choose the leading-order solutions for u_0, v_0 and w_0 as the weighting functions, the first three domain integrals in (70) vanish. Moreover, with $\varphi_1 = u_0$, $\varphi_2 = v_0$ and $\varphi_3 = w_0$, it turns out that the operators L_k ($k = 4, 5, 6, 7, 8, 9$) are related to the stress resultants across the thickness by

$$N_{x(0)} = \int_{-1}^{1} \sigma_{x(0)}\gamma_\beta\,d\eta = L_4(u_0, v_0, w_0), \quad (71)$$

$$N_{xy(0)} = \int_{-1}^{1} \tau_{xy(0)}\gamma_\alpha\,d\eta = L_5(u_0, v_0, w_0), \quad (72)$$

$$N_{yx(0)} = \int_{-1}^{1} \tau_{xy(0)}\gamma_\beta\,d\eta = L_6(u_0, v_0, w_0), \quad (73)$$

$$N_{y(0)} = \int_{-1}^{1} \sigma_{y(0)}\gamma_\alpha\,d\eta = L_7(u_0, v_0, w_0), \quad (74)$$

$$V_{x(0)} = Q_{x(0)} + \frac{\partial M_{yx(0)}}{\partial y} = L_8(u_0, v_0, w_0), \quad (75)$$

$$V_{y(0)} = Q_{y(0)} + \frac{\partial M_{xy(0)}}{\partial x} = L_9(u_0, v_0, w_0), \tag{76}$$

$$Q_{x(0)} = \int_{-1}^{1} \tau_{xz(0)}\,\mathrm{d}\eta, \quad Q_{y(0)} = \int_{-1}^{1} \tau_{yz(0)}\,\mathrm{d}\eta, \tag{77}$$

$$M_{xy(0)} = \int_{-1}^{1} \tau_{xy(0)}\gamma_\alpha \eta\,\mathrm{d}\eta, \quad M_{yx(0)} = \int_{-1}^{1} \tau_{xy(0)}\gamma_\beta \eta\,\mathrm{d}\eta. \tag{78}$$

It follows that the boundary integral terms in (70) merely imply the admissible edge conditions given by (39)–(41) and (44)–(46). The solvability conditions are finally derived from the last remaining term in (70) as

$$u_0 f_{1k} + v_0 f_{2k} + w_0\left(\frac{\partial f_{1k}}{\partial x} + \frac{\partial f_{2k}}{\partial y} - f_{3k}\right) = 0. \tag{79}$$

At the leading-order level (79) is identically satisfied. However, at subsequent levels this imposes an additional condition that has to be satisfied along with the higher-order CST equations. If one makes a straightforward expansion using a single time scale, then all terms in f_{1k}, f_{2k} and f_{3k} are functions of τ_0 only, completely known from the lower-order solution, it will have no room to accommodate the solvability condition. This inevitably leads to inconsistency. As a consequence, the expansion is bound to failure. By constrast, with the multiple scales, f_{1k}, f_{2k} and f_{3k} are yet unknown functions of $\tau_1, \tau_2, \ldots$, in addition to τ_0. This provides enough flexibility in determining the dependence of the field variables upon the scales $\tau_1, \tau_2, \ldots$ and eliminating the secular terms.

6. APPLICATION TO BENCHMARK PROBLEM

For illustration the asymptotic theory is applied to the free vibration problem of doubly curved shells. The problems of isotropic homogeneous shells and cross-ply laminated shells will be considered.

The elastic moduli for orthotropic layers are such that

$$(Q_{16})_i = (Q_{26})_i = (Q_{36})_i = (Q_{45})_i = 0. \tag{80}$$

The boundary conditions on the four edges are of a shear diaphragm type specified by

$$\sigma_\alpha = u_\beta = u_z = 0 \quad \text{on } \alpha = 0 \quad \text{and} \quad \alpha = a_\alpha, \tag{81}$$

$$\sigma_\beta = u_\alpha = u_z = 0 \quad \text{on } \beta = 0 \quad \text{and} \quad \beta = a_\beta. \tag{82}$$

The displacements of various order can be determined by letting

$$u_k = U_k \cos \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \psi), \tag{83}$$

$$v_k = V_k \sin \tilde{m}x \cos \tilde{n}y \cos(\omega\tau_0 - \psi), \tag{84}$$

$$w_k = W_k \sin \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \psi), \tag{85}$$

in which $\tilde{m} = m\pi/a_\alpha$, $\tilde{n} = n\pi/a_\beta$ $(m, n = 1, 2, 3, \ldots)$, and ω denotes the circular frequency of the motion. The phase angle ψ is a function of the time scales $\tau_1, \tau_2, \ldots$ but not τ_0.

Substituting (83)–(85) into (51)–(53), we have the leading-order equations:

$$\begin{bmatrix} \tilde{k}_{11} & \tilde{k}_{12} & \tilde{k}_{13} \\ \tilde{k}_{21} & \tilde{k}_{22} & \tilde{k}_{23} \\ \tilde{k}_{31} & \tilde{k}_{32} & \tilde{k}_{33} \end{bmatrix} \begin{Bmatrix} U_0 \\ V_0 \\ W_0 \end{Bmatrix} = \omega^2 \begin{bmatrix} I_{10} & 0 & -I_{11}\tilde{m} \\ 0 & I_{10} & -I_{11}\tilde{n} \\ -I_{11}\tilde{m} & -I_{11}\tilde{n} & I_{20}+I_{12}(\tilde{m}^2+\tilde{n}^2) \end{bmatrix} \begin{Bmatrix} U_0 \\ V_0 \\ W_0 \end{Bmatrix}, \tag{86}$$

where

$$\tilde{k}_{11} = \tilde{m}^2 \hat{A}_{11} + \tilde{n}^2 \bar{A}_{66},$$

$$\tilde{k}_{12} = \tilde{k}_{21} = \tilde{m}\tilde{n}(\tilde{A}_{12} + \tilde{A}_{66}),$$

$$\tilde{k}_{13} = \tilde{k}_{31} = -\left[\tilde{m}^3 \hat{B}_{11} + \tilde{m}\tilde{n}^2(\bar{B}_{66} + \hat{B}_{12} + \tilde{B}_{66}) + \tilde{m}\left(\frac{\hat{A}_{11}}{R_x} + \frac{\tilde{A}_{12}}{R_y}\right)\right],$$

$$\tilde{k}_{22} = \tilde{m}^2 \hat{A}_{66} + \tilde{n}^2 \bar{A}_{22},$$

$$\tilde{k}_{23} = \tilde{k}_{32} = -\left[\tilde{m}^2\tilde{n}(\hat{B}_{66} + \tilde{B}_{12} + \tilde{B}_{66}) + \tilde{n}^3 \bar{B}_{22} + \tilde{n}\left(\frac{\tilde{A}_{12}}{R_x} + \frac{\bar{A}_{22}}{R_y}\right)\right],$$

$$\tilde{k}_{33} = \tilde{m}^4 \hat{D}_{11} + \tilde{m}^2\tilde{n}^2(2\tilde{D}_{12} + \bar{D}_{66} + 2\tilde{D}_{66} + \hat{D}_{66}) + \tilde{n}^4 \bar{D}_{22}$$
$$+ 2\tilde{m}^2\left(\frac{\hat{B}_{11}}{R_x} + \frac{\tilde{B}_{12}}{R_y}\right) + 2\tilde{n}^2\left(\frac{\tilde{B}_{12}}{R_x} + \frac{\bar{B}_{22}}{R_y}\right) + \left(\frac{\hat{A}_{11}}{R_x^2} + 2\frac{\tilde{A}_{12}}{R_x R_y} + \frac{\bar{A}_{22}}{R_y^2}\right).$$

Equation (86) is an eigenvalue problem, in which the mass and stiffness matrices are real and symmetric, hence ω_i must be real. When the coefficient matrices are positive definite, ω_i are positive. Nontrivial solution of (86) exists if the determinant of the coefficient matrix vanishes.

$$\begin{vmatrix} \tilde{k}_{11} - I_{10}\omega^2 & \tilde{k}_{12} & \tilde{k}_{13} + \tilde{m}I_{11}\omega^2 \\ \tilde{k}_{21} & \tilde{k}_{22} - I_{10}\omega^2 & \tilde{k}_{23} + \tilde{n}I_{11}\omega^2 \\ \tilde{k}_{31} + \tilde{m}I_{11}\omega^2 & \tilde{k}_{32} + \tilde{n}I_{11}\omega^2 & \tilde{k}_{33} - [I_{20} + (\tilde{m}^2 + \tilde{n}^2)I_{12}]\omega^2 \end{vmatrix} = 0. \tag{87}$$

From which three eigenvalues $\omega_i (i = 1, 2, 3)$ that represents the leading-order natural frequencies for a specific set of values of m and n are obtained.

To make the solution unique, the modal displacements are normalized by imposing the orthonormality condition:

$$[(U_0 + \varepsilon^2 U_1 + \varepsilon^4 U_2 + \ldots)\ (V_0 + \varepsilon^2 V_1 + \varepsilon^4 V_2 + \ldots)\ (W_0 + \varepsilon^2 W_1 + \varepsilon^4 W_2 + \ldots)]^T$$
$$\times [(U_0 + \varepsilon^2 U_1 + \varepsilon^4 U_2 + \ldots)\ (V_0 + \varepsilon^2 V_1 + \varepsilon^4 V_2 + \ldots)\ (W_0 + \varepsilon^2 W_1 + \varepsilon^4 W_2 + \ldots)] = 1. \tag{88}$$

Specifically, the normalization conditions at various levels are

$$\varepsilon^0\text{-order:}\quad U_0^2 + V_0^2 + W_0^2 = 1; \tag{89}$$

$$\varepsilon^2\text{-order:}\quad U_0^2 + V_0^2 + W_0^2 = 1,$$
$$U_0 U_1 + V_0 V_1 + W_0 W_1 = 0; \tag{90}$$

$$\varepsilon^4\text{-order:}\quad U_0^2 + V_0^2 + W_0^2 = 1,$$
$$U_0 U_1 + V_0 V_1 + W_0 W_1 = 0,$$
$$U_1^2 + 2U_0 U_2 + V_1^2 + 2V_0 V_2 + W_1^2 + 2W_1 W_2 = 0; \ldots \text{etc.} \tag{91}$$

At the ε^0-order level, the normalized eigenvectors corresponding to $\omega_i (i = 1, 2, 3)$ are written as $[U_0^{(i)}\ V_0^{(i)}\ W_0^{(i)}]^T$. The associated modal stresses and displacements are given in the Appendix.

Corrections to the leading-order solution are determined by carrying on the analysis to the ε^2 order. The ε^2-order solution is governed by the CST equations with non-homogeneous terms given by

$$f_{11}(x,y,1) = \left(\hat{f}_{11}\frac{\partial\psi_i}{\partial\tau_1}+\tilde{f}_{11}\right)\cos\bar{m}x\sin\bar{n}y\cos(\omega_i\tau_0-\psi_i), \quad (92)$$

$$f_{21}(x,y,1) = \left(\hat{f}_{21}\frac{\partial\psi_i}{\partial\tau_1}+\tilde{f}_{21}\right)\cos\bar{m}x\sin\bar{n}y\cos(\omega_i\tau_0-\psi_i), \quad (93)$$

$$f_{31}(x,y,1) = \left(\hat{f}_{31}\frac{\partial\psi_i}{\partial\tau_1}+\tilde{f}_{31}\right)\sin\bar{m}x\sin\bar{n}y\cos(\omega_i\tau_0-\psi_i), \quad (94)$$

where $\hat{f}_{11}, \hat{f}_{21}, \hat{f}_{31}, \tilde{f}_{11}, \tilde{f}_{21}, \tilde{f}_{31}$ are given in the Appendix.

The ε^2-order solution can be determined by letting

$$u_1 = U_1\cos\bar{m}x\sin\bar{n}y\cos(\omega_i\tau_0-\psi_i), \quad (95)$$

$$v_1 = V_1\sin\bar{m}x\cos\bar{n}y\cos(\omega_i\tau_0-\psi_i), \quad (96)$$

$$w_1 = W_1\sin\bar{m}x\sin\bar{n}y\cos(\omega_i\tau_0-\psi_i). \quad (97)$$

Substituting (95)–(97) and (92)–(94) into (60)–(62) gives

$$\begin{bmatrix} \tilde{k}_{11}-I_{10}\omega_i^2 & \tilde{k}_{12} & \tilde{k}_{13}+\bar{m}I_{11}\omega_i^2 \\ \tilde{k}_{12} & \tilde{k}_{22}-I_{10}\omega_i^2 & \tilde{k}_{23}+\bar{n}I_{11}\omega_i^2 \\ \tilde{k}_{13}+\bar{m}I_{11}\omega_i^2 & \tilde{k}_{23}+\bar{n}I_{11}\omega_i^2 & \tilde{k}_{33}-I_{20}\omega_i^2-(\bar{m}^2+\bar{n}^2)I_{12}\omega_i^2 \end{bmatrix}\begin{Bmatrix} U_1 \\ V_1 \\ W_1 \end{Bmatrix}$$
$$= \begin{Bmatrix} \hat{f}_{11}(1)\dfrac{\partial\psi_i}{\partial\tau_1}+\tilde{f}_{11}(1) \\ \hat{f}_{21}(1)\dfrac{\partial\psi_i}{\partial\tau_1}+\tilde{f}_{21}(1) \\ [\hat{f}_{31}(1)+\bar{m}\hat{f}_{11}(1)+\bar{n}\hat{f}_{21}(1)]\dfrac{\partial\psi_i}{\partial\tau_1}+[\tilde{f}_{31}(1)+\bar{m}\tilde{f}_{11}(1)+\bar{n}\tilde{f}_{21}(1)] \end{Bmatrix}. \quad (98)$$

Equation (98) is solvable if and only if the solvability condition (79) is satisfied. The dependence of ψ_i upon τ_1 can then be determined as

$$\psi_i = -\lambda_i\tau_1+\tilde{\psi}_i(\tau_2,\tau_3,\ldots), \quad (99)$$

where

$$\lambda_i = \frac{U_0^{(i)}\tilde{f}_{11}(1)+V_0^{(i)}\tilde{f}_{21}(1)-W_0^{(i)}[\tilde{f}_{31}(1)+\bar{m}\tilde{f}_{11}(1)+\bar{n}\tilde{f}_{21}(1)]}{U_0^{(i)}\hat{f}_{11}(1)+V_0^{(i)}\hat{f}_{21}(1)-W_0^{(i)}[\hat{f}_{31}(1)+\bar{m}\hat{f}_{11}(1)+\bar{n}\hat{f}_{21}(1)]},$$

and $\tilde{\psi}_i$ represents integration functions of the scales $\tau_2, \tau_3, \ldots$, and so on.

With (99) and the relation $\tau_1 = \varepsilon^2\tau_0 = h/R\tau_0$, the time functions of all field variables are now expressed in terms of $\cos[(\omega+\lambda h/R)\tau_0-\tilde{\psi}]$. Therefore the natural frequencies at the ε^2-order level have been modified to

$$\omega_i + \lambda_i \frac{h}{R} \quad (i = 1, 2, 3). \tag{100}$$

Substituting (99) into (98), we obtain

$$\begin{bmatrix} \tilde{k}_{11} - I_{10}\omega_i^2 & \tilde{k}_{12} & \tilde{k}_{13} + \tilde{m} I_{11}\omega_i^2 \\ \tilde{k}_{12} & \tilde{k}_{22} - I_{10}\omega_i^2 & \tilde{k}_{23} + \tilde{n} I_{11}\omega_i^2 \\ \tilde{k}_{13} + \tilde{m} I_{11}\omega_i^2 & \tilde{k}_{23} + \tilde{n} I_{11}\omega_i^2 & \tilde{k}_{33} - I_{20}\omega_i^2 - (\tilde{m}^2 + \tilde{n}^2) I_{12}\omega_i^2 \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ W_1 \end{Bmatrix}$$

$$= \begin{Bmatrix} -\lambda_i \hat{f}_{11}(1) + \tilde{f}_{11}(1) \\ -\lambda_i \hat{f}_{21}(1) + \tilde{f}_{21}(1) \\ -\lambda_i [\hat{f}_{31}(1) + \tilde{m}\hat{f}_{11}(1) + \tilde{n}\hat{f}_{21}(1)] + [\tilde{f}_{31}(1) + \tilde{m}\tilde{f}_{11}(1) + \tilde{n}\tilde{f}_{21}(1)] \end{Bmatrix}, \tag{101}$$

from which U_1, V_1, W_1 can be uniquely determined by a simple solution of the algebraic equations along with the normalization conditions (90).

Once U_1, V_1, W_1 are obtained, determination of the ε^2-order corrections of the modal stresses and displacements is straightforward. The solution procedure can be continued to higher levels in a similar way.

7. COMPARISON OF RESULTS

Orthotropic laminated plates

The present analysis is applicable to laminated plates simply by letting $1/R_\alpha = 1/R_\beta = 0$ in the formulation. For comparison, the free vibration problem of the simply supported three-ply orthotropic laminated plates considered in Srinivas and Rao (1970) is computed. In the computations, the ratios of c_{ij}/c_{11} are taken to be the same for each layer. The data are

$$c_{12}/c_{11} = 0.233190, \quad c_{13}/c_{11} = 0.010776, \quad c_{22}/c_{11} = 0.543103,$$
$$c_{23}/c_{11} = 0.098276, \quad c_{33}/c_{11} = 0.530172, \quad c_{44}/c_{11} = 0.266810,$$
$$c_{55}/c_{11} = 0.159914, \quad c_{66}/c_{11} = 0.262931.$$

The material of top and bottom layers of the laminate is identical. The geometry parameters are $a_\alpha = a_\beta$, $a_\alpha/2h = 10$ and $h_1 : h_2 : h_3 = 0.1 : 0.8 : 0.1$. The asymptotic solutions are compared with the elasticity solutions (Srinivas and Rao, 1970), the FSDT solutions (Whitney and Pagano, 1970) and the classical laminated plate solutions (CPT). Table 1 shows the dimensionless fundamental frequency Ω ($\Omega = \omega\sqrt{2\rho h/(c_{11})_2}$) for five different ratios of $(c_{11})_1/(c_{11})_2$ which indicates the ratio of material anisotropy between the layers. The convergence of the present solution in the case of $(c_{11})_1/(c_{11})_2 = 1$ is more rapid than $(c_{11})_1/(c_{11})_2 = 15$. The asymptotic analysis yields results in close agreement with the elasticity solution. In the cases of a small difference in layer properties (for example, $(c_{11})_1/(c_{11})_2 = 1, 2$), the results after two steps are quite acceptable. When a large difference in the layer properties is involved (for example, $(c_{11})_1/(c_{11})_2 = 15$), it is necessary to carry out the analysis to higher levels to obtain accurate results. The FSDT results vary with different assumed values of the shear correction factor k. In the present analysis the correction factor is not required.

Cylindrical and spherical laminated shells

The cylindrical shell and the spherical shell are special cases of doubly curved shells in which $1/R_\alpha = 0$ and $R_\alpha = R_\beta$, respectively. The dynamic responses of these type of shells composed of orthotropic layers with [0°/90°] construction are computed. The material properties are given as $E_1 = 25E_2$, $E_3 = E_2$, $G_{13} = G_{12} = 0.5E_2$, $G_{23} = 0.2E_2$, $\nu_{12} = 0.25$, $\nu_{31} = 0.03$, $\nu_{23} = 0.4$. The dimensionless frequency Ω is defined as $\Omega = \omega a_\alpha\sqrt{\rho/E_2}$. Comparisons among the asymptotic solution, the approximate elasticity solution based on a

Table 1. Fundamental frequencies (Ω) of orthotropic homogeneous plates ($\Omega = \omega a_\alpha \sqrt{\rho / E_2}$)

$(C_{11})_1/(C_{11})_2$	Present study		Srinivas-Rao (1970) Elasticity sol.	Whitney-Pagano (1970) FSDT			CPT
				$k = 1$	$k = 2/3$	$k = 5/6$	
1	ε^0	0.049262	0.047419	0.047698	0.046971	0.047403	0.049666
	ε^2	0.047304					
	ε^4	0.047427					
	ε^6	0.047418					
2	ε^0	0.060092	0.057041	0.057753	0.056685	0.057318	0.060584
	ε^2	0.056774					
	ε^4	0.057063					
	ε^6	0.057035					
5	ε^0	0.084640	0.077148	0.080364	0.078465	0.079587	0.085333
	ε^2	0.075960					
	ε^4	0.077356					
	ε^6	0.077111					
10	ε^0	0.114391	0.098104	0.107696	0.104777	0.106498	0.115328
	ε^2	0.093569					
	ε^4	0.099502					
	ε^6	0.097660					
15	ε^0	0.137864	0.112034	0.129283	0.125574	0.127758	0.138994
	ε^2	0.101795					
	ε^4	0.116559					
	ε^6	0.109975					

discrete-layer approach (Bhimaraddi, 1991) and various solutions based on displacement models (Bhimaraddi, 1984; Leissa and Qatu, 1991) are presented in Tables 2 and 3. The ε^0-order solution agrees well with the CST solution in the case of thin shells. In the cases of moderately thick shells the CST solution begins to deteriorate. The ε^2-order solution converges in the case of thin shells ($2h/a_\alpha = 0.05$), the ε^4-order solution in the case of moderately thick shells ($2h/a_\alpha = 0.1$), and the ε^6-order solution in the case of thick shells ($2h/a_\alpha = 0.15$). Heterogeneity effects of the transverse deformation become significant for thick shells. The number of sublayers needed to be taken in the approximate elasticity solution (Bhimaraddi, 1991) is estimated to be 6, 10 and 16 in the cases of $2h/a_\alpha = 0.05$, 0.1 and 0.15, respectively.

Doubly curved laminated shells. Numerical results on doubly curved laminated shells available for comparison purpose can hardly be found. The present solution may serve as a benchmark in assessing the applicability of various two-dimensional shell theories. In the computations the layer material properties are taken to be $E_1 = 25E_2$, $E_3 = E_2$, $G_{13} = G_{12} = 0.5E_2$, $G_{23} = 0.2E_2$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$. The asymptotic solution has been computed to the ε^6 order. The results of [0°/90°] laminated shells are shown in Table 4. Convergence is found to be fast. In fact, the solution for laminates with $R_\alpha/a_\alpha = 1$, $R_\beta/a_\beta = 5$ and $a_\alpha = a_\beta$ is convergent at ε^2 order for the thin shells ($2h/a_\alpha = 0.05$), at ε^4 order for the moderately thick shells ($2h/a_\alpha = 0.1$), and at ε^6 order for the thick shells ($2h/a_\alpha = 0.15$). The modal displacement and stress distributions through the thickness direction at various levels for laminates with $2h/a_\alpha = 0.1$ are shown in Figs 2–6. Indeed rapid convergent interlaminar stresses corresponding to the modal displacements are obtained without treating the system layer by layer. Table 5 contains the results of ten-layer symmetric and antisymmetric shells ($[0°/90°/0°/90°/0°]_2$ and $[0°/90°]_5$). Table 6 presents the fundamental and higher frequencies for various [0°/90°] laminated shells with $2h/a_\alpha = 0.05$. Both of the tables reveal the convergent solutions are obtained with fast speed.

8. CONCLUSIONS

By means of the method of multiple time scales an asymptotic theory is developed for the dynamic analysis of the doubly curved laminated shells. The asymptotic theory is based

Table 2. Fundamental frequencies (Ω) of [0°/90°] cylindrical shells ($\Omega = \omega a_\alpha \sqrt{\rho/E_2}$)

$2h/a_\alpha$	R_β/a_β	Present study	Bhimaraddi (1991) Elasticity sol.	Bhimaraddi (1984) HSDT	Bhimaraddi (1984) FSDT($k = \pi^2/12$)	Leissa-Qatu (1991) CST
0.05	1	ε^0 0.80556 ε^2 0.79235 ε^4 0.79305 ε^6 0.79307	0.78683	0.79993	0.79798	0.80580
0.05	5	ε^0 0.50233 ε^2 0.49307 ε^4 0.49332 ε^6 0.49331	0.49167	0.49402	0.49091	0.50216
0.05	10	ε^0 0.48838 ε^2 0.47923 ε^4 0.47949 ε^6 0.47948	0.47859	0.47997	0.47677	0.48827
0.10	1	ε^0 1.14432 ε^2 1.06117 ε^4 1.06901 ε^6 1.06875	1.04085	1.09189	1.07475	1.14313
0.10	5	ε^0 0.96953 ε^2 0.89914 ε^4 0.90659 ε^6 0.90573	0.90200	0.90953	0.88840	0.96870
0.10	10	ε^0 0.96120 ε^2 0.89067 ε^4 0.89829 ε^6 0.89740	0.89564	0.90150	0.88026	0.96074
0.15	1	ε^0 1.54666 ε^2 1.30110 ε^4 1.34880 ε^6 1.34070	1.29099	1.38174	1.33274	1.54124
0.15	5	ε^0 1.42657 ε^2 1.20437 ε^4 1.25336 ε^6 1.24210	1.23849	1.25551	1.20020	1.42464
0.15	10	ε^0 1.41810 ε^2 1.19613 ε^4 1.24558 ε^6 1.23413	1.23374	1.24875	1.19342	1.41709

Table 3. Fundamental frequencies (Ω) of [0°/90°] spherical shells ($\Omega = \omega a_\alpha \sqrt{\rho/E_2}$)

$2h/a_\alpha$	R_β/a_β	Present study	Bhimaraddi (1991) Elasticity sol.	Bhimaraddi (1984) HSDT	Bhimaraddi (1984) FSDT ($k = \pi^2/12$)	Leissa-Qatu (1991) CST
0.05	1	ε^0 1.32988 ε^2 1.30937 ε^4 1.31073 ε^6 1.31072	1.29835	1.32595	1.32483	1.33000
0.05	1	ε^0 0.55246 ε^2 0.54221 ε^4 0.54250 ε^6 0.54249	0.54039	0.54500	0.54219	0.55247
0.05	10	ε^0 0.50149 ε^2 0.49193 ε^4 0.49220 ε^6 0.49219	0.49127	0.49341	0.49031	0.50149
0.10	1	ε^0 1.52339 ε^2 1.42927 ε^4 1.44079 ε^6 1.43995	1.39974	1.49075	1.48008	1.52391
0.10	5	ε^0 0.99032 ε^2 0.91785 ε^4 0.92564 ε^6 0.92473	0.92065	0.93361	0.91338	0.99034
0.10	10	ε^0 0.96519 ε^2 0.89383 ε^4 0.90157 ε^6 0.90066	0.89912	0.90679	0.88584	0.96519
0.15	1	ε^0 1.78806 ε^2 1.54523 ε^4 1.59683 ε^6 1.58762	1.51936	1.68141	1.64797	1.78940
0.15	5	ε^0 1.43115 ε^2 1.20829 ε^4 1.25737 ε^6 1.24612	1.24272	1.26797	1.21434	1.43120
0.15	10	ε^0 1.41638 ε^2 1.19447 ε^4 1.24377 ε^6 1.23240	1.23249	1.25034	1.19559	1.41639

Table 4. Fundamental frequencies (Ω) of [0°/90°] doubly curved shells ($\Omega = \omega a_\alpha \sqrt{\rho/E_2}$)

$2h/a_\alpha$	R_α/a_α	R_β/a_β	Present study			
			ε^0	ε^2	ε^4	ε^6
0.05	1	1	1.3299	1.3098	1.3111	1.3111
	1	5	0.8967	0.8723	0.8735	0.8735
	1	10	0.8413	0.8160	0.8173	0.8173
	1	20	0.8139	0.7881	0.7895	0.7894
	−1	1	0.4446	0.3961	0.3962	0.3963
	−1	5	0.7016	0.6853	0.6861	0.6861
	−1	10	0.7528	0.7383	0.7390	0.7390
	−1	20	0.7790	0.7651	0.7659	0.7659
0.10	1	1	1.5234	1.4306	1.4415	1.4409
	1	5	1.1722	1.0717	1.0823	1.0826
	1	10	1.1299	1.0271	1.0379	1.0383
	1	20	1.1093	1.0052	1.0159	1.0164
	−1	1	0.9053	0.7638	0.7711	0.7729
	−1	5	1.0751	0.9846	0.9932	0.9928
	−1	10	1.1089	1.0218	1.0305	1.0299
	−1	20	1.1264	1.0408	1.0494	1.0488
0.15	1	1	1.7881	1.5469	1.5977	1.5886
	1	5	1.5031	1.2516	1.3062	1.2958
	1	10	1.4688	1.2138	1.2691	1.2586
	1	20	1.4520	1.1950	1.2506	1.2401
	−1	1	1.3752	1.0405	1.0947	1.1033
	−1	5	1.5016	1.2413	1.2943	1.2847
	−1	10	1.5237	1.2694	1.3221	1.3118
	−1	20	1.5351	1.2834	1.3360	1.3255

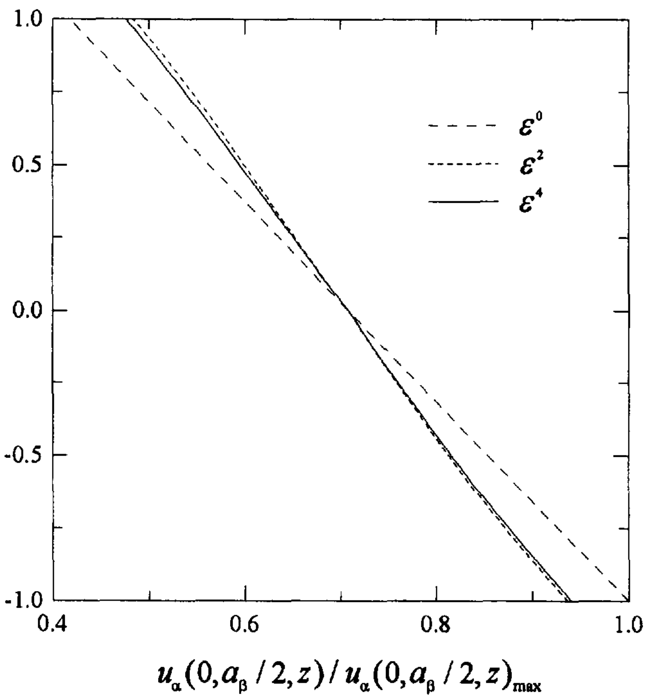

Fig. 2. The distribution of the modal displacement through the thickness of the [0°/90°] doubly curved shell.

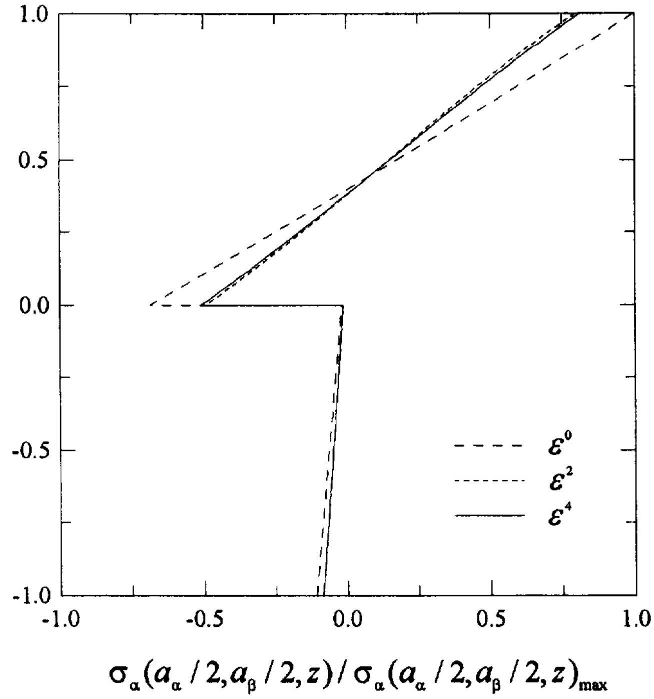

Fig. 3. The distribution of the modal membrane stress through the thickness of the [0°/90°] doubly curved shell.

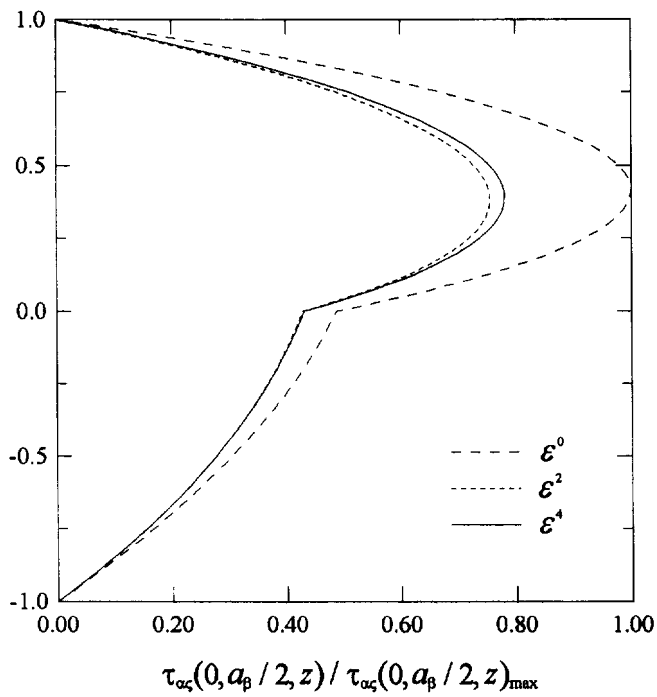

Fig. 4. The distribution of the modal transverse shear stress through the thickness of the [0°/90°] doubly curved shell.

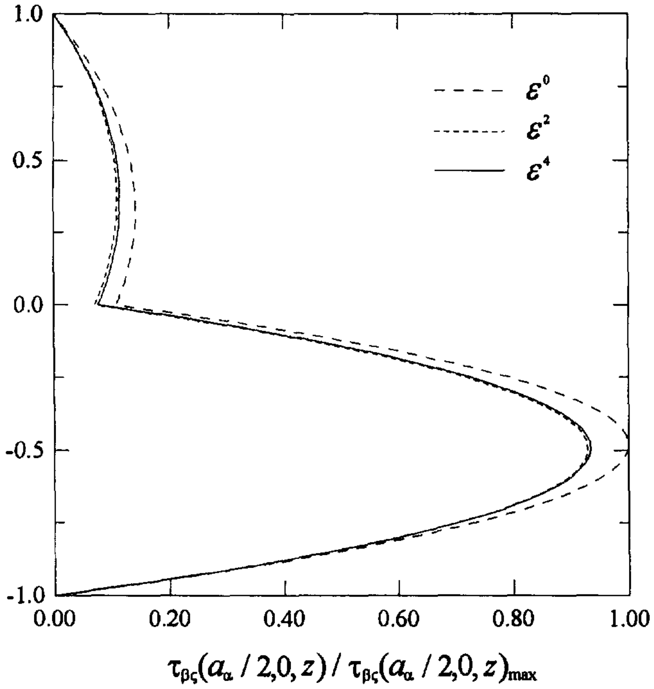

Fig. 5. The distribution of the modal transverse shear stress through the thickness of the [0°/90°] doubly curved shell.

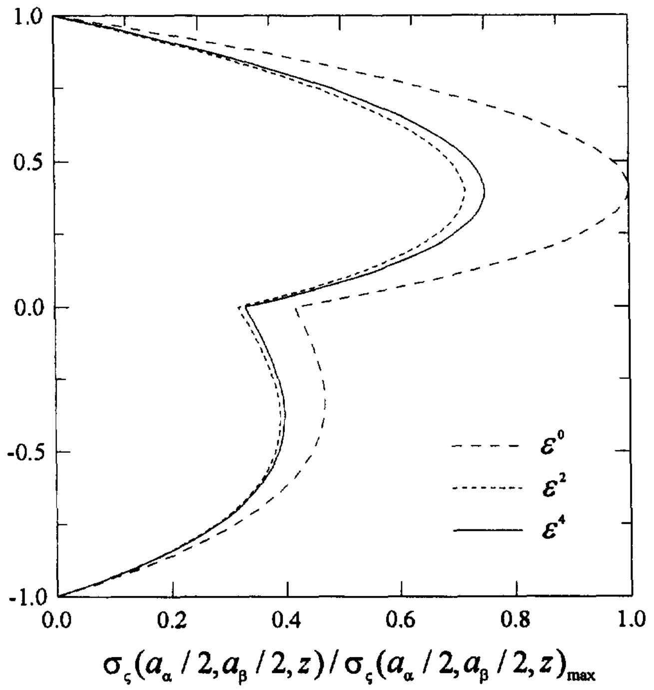

Fig. 6. The distribution of the modal transverse normal stress through the thickness of the [0°/90°] doubly curved shell.

Table 5. Fundamental frequencies (Ω) of [0°/90°]$_s$ and [0°/90°/0°/90°/0°]$_2$ doubly curved shells ($\Omega = \omega a_\alpha \sqrt{\rho/E_2}$)

Laminates	$2h/a_\alpha$	R_α/a_α	R_β/a_β	Present study ε^0	ε^2	ε^4	ε^6
[0°/90°]$_s$	0.05	1	1	1.4324	1.3798	1.3859	1.3856
		1	5	1.0568	1.0012	1.0073	1.0068
		1	10	1.0103	0.9527	0.9590	0.9585
		1	20	0.9874	0.9287	0.9350	0.9346
		−1	1	0.6845	0.5744	0.5820	0.5823
		−1	5	0.8855	0.8212	0.8278	0.8274
		−1	10	0.9269	0.8662	0.8726	0.8721
		−1	20	0.9483	0.8893	0.8955	0.8951
	0.10	1	1	1.8558	1.5172	1.6403	1.5987
		1	5	1.6140	1.2517	1.3814	1.3335
		1	10	1.5841	1.2163	1.3474	1.2992
		1	20	1.5695	1.1985	1.3304	1.2822
		−1	1	1.3718	0.9030	1.0541	1.0179
		−1	5	1.5219	1.1300	1.2683	1.2191
		−1	10	1.5474	1.1635	1.3001	1.2505
		−1	20	1.5606	1.1804	1.3161	1.2665
[0°/90°/0°/90°/0°]$_2$	0.05	1	1	1.4357	1.3813	1.3878	1.3874
		1	5	1.0638	1.0015	1.0085	1.0080
		1	10	1.0178	0.9530	0.9603	0.9598
		1	20	0.9952	0.9291	0.9364	0.9359
		−1	1	0.6887	0.5752	0.5835	0.5834
		−1	5	0.8885	0.8139	0.8217	0.8212
		−1	10	0.9297	0.8588	0.8665	0.8659
		−1	20	0.9511	0.8819	0.8895	0.8889
	0.10	1	1	1.8680	1.5168	1.6489	1.6020
		1	5	1.6360	1.2444	1.3890	1.3342
		1	10	1.6074	1.2091	1.3553	1.3001
		1	20	1.5933	1.1915	1.3386	1.2832
		−1	1	1.3721	0.8930	1.0517	1.0035
		−1	5	1.5262	1.1062	1.2570	1.2013
		−1	10	1.5523	1.1397	1.2891	1.2334
		−1	20	1.5658	1.1568	1.3055	1.2498

Table 6. Natural frequencies (Ω) of [0°/90°] doubly curved shells ($\Omega = \omega a_\alpha \sqrt{\rho/E_2}$)

R_α/a_α	R_β/a_β	(m, n)	Present study ε^0	ε^2	ε^4	ε^6
		(1,1)	1.3299	1.3098	1.3111	1.3111
		(1,2)	2.0168	1.9395	1.9444	1.9442
		(1,3)	3.3721	3.0513	3.1100	3.0980
		(2,1)	1.9986	1.9117	1.9227	1.9214
1	1	(2,2)	2.2899	2.1322	2.1488	2.1470
		(2,3)	3.4806	3.0706	3.1436	3.1294
		(3,1)	3.3248	2.9529	3.0424	3.0203
		(3,2)	3.4662	3.0000	3.1026	3.0777
		(3,3)	4.3949	3.7177	3.8654	3.8319
		(1,1)	0.8967	0.8723	0.8735	0.8735
		(1,2)	1.8267	1.7556	1.7613	1.7608
		(1,3)	3.2599	2.9454	3.0086	2.9945
		(2,1)	1.4144	1.2967	1.3082	1.3070
1	5	(2,2)	2.0296	1.8672	1.8849	1.8829
		(2,3)	3.3304	2.9160	2.9957	2.9787
		(3,1)	2.8188	2.3837	2.4747	2.4538
		(3,2)	3.2278	2.7505	2.8507	2.8276
		(3,3)	4.2392	3.5577	3.7087	3.6739
		(1,1)	0.8413	0.8160	0.8173	0.8173
		(1,2)	1.8021	1.7304	1.7363	1.7357
		(1,3)	3.2448	2.9292	2.9932	2.9788
		(2,1)	1.3679	1.2479	1.2592	1.2581
1	10	(2,2)	2.0023	1.8383	1.8562	1.8542
		(2,3)	3.3123	2.8955	2.9762	2.9589
		(3,1)	2.7900	2.3534	2.4437	2.4232
		(3,2)	3.2069	2.7292	2.8288	2.8059
		(3,3)	4.2221	3.5390	3.6905	3.6555
		(1,1)	0.8139	0.7881	0.7894	0.7894
		(1,2)	1.7897	1.7176	1.7236	1.7231
		(1,3)	3.2372	2.9208	2.9852	2.9708
		(2,1)	1.3482	1.2272	1.2385	1.2374
1	20	(2,2)	1.9891	1.8243	1.8423	1.8402
		(2,3)	3.3033	2.8852	2.9664	2.9489
		(3,1)	2.7792	2.3424	2.4324	2.4120
		(3,2)	3.1973	2.7194	2.8187	2.7960
		(3,3)	4.2138	3.5298	3.6816	3.6465

on the three-dimensional elasticity theory without making static and kinematic assumptions *a priori*. The leading-order equations for the asymptotic analysis are the CST equations. The higher-order equations are identical to the leading-order equations except for non-homogeneous terms that may produce secular terms in the asymptotic solution. Solvability conditions corresponding to the higher-order problems are derived to eliminate the secular terms in a simple and systematic way. Application of the present theory to the laminated shells reveals that the asymptotic solution is convergent to the elasticity solution available in the literature at ε^2 order for the thin shells ($2h/a_\alpha = 0.05$), at ε^4 order for the moderately thick shells ($2h/a_\alpha = 0.1$), and at ε^6 order for the thick shells ($2h/a_\alpha = 0.15$).

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REFERENCES

Agalovian, L. A. (1966). Application of the method of asymptotic integration to the construction of an approximate theory of anisotropic shells. *PMM* **30,** 388–398.

Bhimaraddi, A. (1984). A higher-order theory for free vibration analysis of circular cylindrical shells. *Int. J. Solids Struct.* **20,** 623–630.

Bhimaraddi, A. (1991). Free vibration analysis of doubly curved shallow shells on rectangular planform using three-dimensional elasticity theory. *Int. J. Solids Struct.* **27,** 897–913.

Bhimaraddi, A. (1993). Three-dimensional elasticity solution for static response of orthotropic doubly curved shallow shells on rectangular planform. *Comp. Struct.* **24,** 67–77.

Fan, J. and Zhang, J. (1992). Analytical solutions for thick, doubly curved, laminated shells. *J. Engng Mech. ASCE* **118,** 1338–1356.
Huang, N. N. (1995). Exact analysis for three-dimensional free vibrations of cross-ply cylindrical and doubly-curved laminates. *Acta Mech.* **108,** 23–34.
Huang, N. N. and Tauchert, T. R. (1992). Thermal stresses in doubly-curved cross-ply laminates. *Int. J. Solids Struct.* **29,** 991–1000.
Johnson, M. W. and Widera, O. E. (1971). An asymptotic theory for the vibration of non-homogeneous plates. *Acta Mech.* **12,** 121–142.
Kapania, R. K. (1989). A review on the analysis of laminated shells. *J. Pressure Vessel Tech.* **111,** 88–96.
Khdeir, A. A., Librescu, L. and Frederick, D. (1989). A shear deformable theory of laminated composite shallow shell-type panels and their response analysis II: static response. *Acta Mech.* **77,** 1–12.
Leissa, A. W. and Qatu, M. S. (1991). Equations of elastic deformation of laminated composite shallow shells. *J. Appl. Mech.* **58,** 181–188.
Librescu, L., Khdeir, A. A. and Frederick, D. (1989). A shear deformable theory of laminated composite shallow shell-type panels and their response analysis. I: free vibration and buckling. *Acta Mech.* **76,** 1–33.
Logan, D. L. and Widera, G. E. O. (1980). Refined theories for nonhomogeneous anisotropic cylindrical shells: part II—application. *J. Engng Mech. ASCE* **106,** 1075–1090.
Nayfeh, A. H. (1981). *Introduction to Perturbation Techniques.* Wiley, New York.
Noor, A. K. and Burton, W. S. (1990). Assessment of computational models for multilayered composite shells. *Appl. Mech. Rev. ASME* **43,** 67–97.
Reddy, J. N. (1984). Exact solutions of moderatively thick laminated shells. *J. Engng Mech. ASCE* **110,** 794–808.
Reddy, J. N. and Liu, C. F. (1985). A higher-order shear deformation theory of laminated elastic shells. *Int. J. Engng Sci.* **23,** 319–330.
Srinivas, S. and Rao, A. K. (1970). Bending, vibration and buckling of simply supported thick orthotropic plates and laminates. *Int. J. Solids Struct.* **6,** 1463–1481.
Tarn, J. Q. (1994). An asymptotic theory for dynamic response of anisotropic inhomogeneous and laminated cylindrical shells. *J. Mech. Phys. Solids* **42,** 1633–1650.
Tarn, J. Q. and Wang, Y. M. (1994). An asymptotic theory for dynamic response of anisotropic inhomogeneous and laminated plates. *Int. J. Solids. Struct.* **31,** 231–246.
Tarn, J. Q. and Yen, C. B. (1995) A three-dimensional asymptotic analysis of anisotropic inhomogeneous and laminated cylindrical shells. *Acta Mech.* (in press).
Whitney, J. M. and Pagano, N. J. (1970). Shear deformation in heterogeneous anisotropic plates. *J. Appl. Mech.* **37,** 1031–1036.
Widera, O. E. (1970). An asymptotic theory for the motion of elastic plates. *Acta Mech.* **9,** 54–66.
Widera, G. E. O. and Logan, D. L. (1980). Refined theories for nonhomogeneous anisotropic cylindrical shells: part I—derivation. *J. Engng Mech. ASCE* **106,** 1053–1074.

APPENDIX

The ε^{2k}-order solution ($k = 0, 1, 2, \ldots$) for the displacements and stresses in the illustrative problem are given by

$$u_{(k)} = \tilde{u}_{(k)} \cos \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \psi), \tag{A1}$$

$$v_{(k)} = \tilde{v}_{(k)} \sin \tilde{m}x \cos \tilde{n}y \cos(\omega\tau_0 - \psi), \tag{A2}$$

$$w_{(k)} = \tilde{w}_{(k)} \sin \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \psi), \tag{A3}$$

$$\sigma_{x(k)} = \tilde{\sigma}_{x(k)} \sin \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \psi), \tag{A4}$$

$$\sigma_{y(k)} = \tilde{\sigma}_{y(k)} \sin \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \psi), \tag{A5}$$

$$\sigma_{z(k)} = \tilde{\sigma}_{z(k)} \sin \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \psi), \tag{A6}$$

$$\tau_{xz(k)} = \tilde{\tau}_{xz(k)} \cos \tilde{m}x \sin \tilde{n}y \cos(\omega\tau_0 - \psi), \tag{A7}$$

$$\tau_{yz(k)} = \tilde{\tau}_{yz(k)} \sin \tilde{m}x \cos \tilde{n}y \cos(\omega\tau_0 - \psi), \tag{A8}$$

$$\tau_{xy(k)} = \tilde{\tau}_{xy(k)} \cos \tilde{m}x \cos \tilde{n}y \cos(\omega\tau_0 - \psi), \tag{A9}$$

where, at the ε^0-order level,

$$\tilde{u}_{(0)} = U_{0(mn)} - z\tilde{m}W_{0(mn)},$$

$$\tilde{v}_{(0)} = V_{0(mn)} - z\tilde{n}W_{0(mn)},$$

$$\tilde{w}_{(0)} = W_{0(mn)},$$

$$\tilde{\sigma}_{x(0)} = -\frac{\tilde{Q}_{11}\tilde{m}}{\gamma_\alpha} U_{0(mn)} - \frac{\tilde{Q}_{12}\tilde{n}}{\gamma_\beta} V_{0(mn)} + \left(\frac{\tilde{Q}_{11}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{12}}{\gamma_\beta R_y}\right) W_{0(mn)},$$

$$\tilde{\sigma}_{y(0)} = -\frac{\tilde{Q}_{12}\tilde{m}}{\gamma_\alpha} U_{0(mn)} - \frac{\tilde{Q}_{22}\tilde{n}}{\gamma_\beta} V_{0(mn)} + \left(\frac{\tilde{Q}_{12}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{22}}{\gamma_\beta R_y}\right) W_{0(mn)},$$

$$\tilde{\tau}_{xy(0)} = \frac{\tilde{Q}_{66}\tilde{n}}{\gamma_\beta} U_{0(mn)} + \frac{\tilde{Q}_{66}\tilde{m}}{\gamma_\alpha} V_{0(mn)},$$

$$\tilde{\tau}_{xz(0)} = \int_{-1}^{z} \left[\left(\frac{\tilde{m}^2 \tilde{Q}_{11}\gamma_\beta}{\gamma_\alpha} + \frac{\tilde{n}^2 \tilde{Q}_{66}\gamma_\alpha}{\gamma_\beta} \right) \tilde{u}_{(0)} + \tilde{m}\tilde{n}(\tilde{Q}_{12} + \tilde{Q}_{66})\tilde{v}_{(0)} - \tilde{m}\left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x \gamma_\alpha} + \frac{\tilde{Q}_{12}}{R_y} \right) \tilde{w}_{(0)} - \rho_1 \omega^2 \tilde{u}_{(0)} \right] d\eta,$$

$$\tilde{\tau}_{yz(0)} = \int_{-1}^{z} \left[\tilde{m}\tilde{n}(\tilde{Q}_{12} + \tilde{Q}_{66})\tilde{u}_{(0)} + \left(\frac{\tilde{m}^2 \tilde{Q}_{66}\gamma_\beta}{\gamma_\alpha} + \frac{\tilde{n}^2 \tilde{Q}_{22}\gamma_\alpha}{\gamma_\beta} \right) \tilde{v}_{(0)} - \tilde{n}\left(\frac{\tilde{Q}_{12}}{R_x} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y \gamma_\beta} \right) \tilde{w}_{(0)} - \rho_1 \omega^2 \tilde{v}_{(0)} \right] d\eta,$$

$$\tilde{\sigma}_{z(0)} = \int_{-1}^{z} \left[-\tilde{m}\left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha} + \frac{\tilde{Q}_{12}}{R_y} \right)\tilde{u}_{(0)} - \tilde{n}\left(\frac{\tilde{Q}_{12}}{R_x} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta} \right)\tilde{v}_{(0)} + \left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x^2\gamma_\alpha} + \frac{2\tilde{Q}_{12}}{R_xR_y} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y^2\gamma_\beta} \right)\tilde{w}_{(0)} + \tilde{m}\tilde{\tau}_{xz(0)} + \tilde{n}\tilde{\tau}_{yz(0)} - \rho_2\omega^2\tilde{w}_{(0)} \right] d\eta.$$

The expressions of $\hat{f}_{11}, \hat{f}_{21}, \hat{f}_{31}, \tilde{f}_{11}, \tilde{f}_{21}, \tilde{f}_{31}$ and the revelant functions for the ε^2-order corrections are

$$\hat{f}_{11} = \int_{-1}^{z} 2\rho_1 \omega \tilde{u}_{(0)}\, d\eta, \tag{A10}$$

$$\hat{f}_{21} = \int_{-1}^{z} 2\rho_1 \omega \tilde{v}_{(0)}\, d\eta, \tag{A11}$$

$$\hat{f}_{31} = \int_{-1}^{z} (2\rho_2 \omega \tilde{w}_{(0)} - \tilde{m}\hat{f}_{11} - \tilde{n}\hat{f}_{21})\, d\eta, \tag{A12}$$

$$\tilde{f}_{11}(z) = \int_{-1}^{z} \left[\left(\frac{\tilde{m}^2 \tilde{Q}_{11}\gamma_\beta}{\gamma_\alpha} + \frac{\tilde{n}^2 \tilde{Q}_{66}\gamma_\alpha}{\gamma_\beta} \right)\tilde{\phi}_{11} + \tilde{m}\tilde{n}(\tilde{Q}_{12} + \tilde{Q}_{66})\tilde{\phi}_{21} - \tilde{m}\left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha} + \frac{\tilde{Q}_{12}}{R_y} \right)\tilde{\phi}_{31} - \frac{1}{R_x}\tilde{\tau}_{xz(0)} - \tilde{m}\tilde{c}_{13}\gamma_\beta\tilde{\sigma}_{z(0)} - \rho\omega^2\tilde{\phi}_{11} \right] d\eta - \left(\frac{1}{R_x} + \frac{1}{R_y} \right) z\tilde{\tau}_{xz(0)}, \tag{A13}$$

$$\tilde{f}_{21}(z) = \int_{-1}^{z} \left[\tilde{m}\tilde{n}(\tilde{Q}_{12} + \tilde{Q}_{66})\tilde{\phi}_{11} + \left(\frac{\tilde{m}^2 \tilde{Q}_{66}\gamma_\beta}{\gamma_\alpha} + \frac{\tilde{n}^2 \tilde{Q}_{22}\gamma_\alpha}{\gamma_\beta} \right)\tilde{\phi}_{21} - \tilde{n}\left(\frac{\tilde{Q}_{12}}{R_x} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta} \right)\tilde{\phi}_{31} - \frac{1}{R_y}\tilde{\tau}_{yz(0)} - \tilde{n}\tilde{c}_{23}\gamma_\alpha\tilde{\sigma}_{z(0)} - \rho_1\omega^2\tilde{\phi}_{21} \right] d\eta - \left(\frac{1}{R_x} + \frac{1}{R_y} \right) z\tilde{\tau}_{yz(0)}, \tag{A14}$$

$$\tilde{f}_{31}(z) = \left(\frac{1}{R_x} + \frac{1}{R_y} \right) z\tilde{\sigma}_{z(0)} + \int_{-1}^{z} \left\{ -\left(\frac{\tilde{c}_{13}}{R_x} + \frac{\tilde{c}_{23}}{R_y} \right)\tilde{\sigma}_{z(0)} + \tilde{m}\left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha} + \frac{\tilde{Q}_{12}}{R_y} \right)\tilde{\phi}_{11} + \tilde{n}\left(\frac{\tilde{Q}_{12}}{R_x} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta} \right)\tilde{\phi}_{21} - \left(\frac{Q_{11}\gamma_\beta}{R_x^2\gamma_\alpha} + \frac{2Q_{12}}{R_xR_y} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y^2\gamma_\beta} \right)\tilde{\phi}_{31} - \tilde{m}\tilde{f}_{11} - \tilde{n}\tilde{f}_{21} - \left(\frac{\tilde{m}\eta}{R_y} \right)\tilde{\tau}_{xz(0)} - \left(\frac{\tilde{n}\eta}{R_x} \right)\tilde{\tau}_{yz(0)} - \rho_2\omega^2\tilde{\phi}_{31} \right\} d\eta, \tag{A15}$$

$$\tilde{\phi}_{11}(z) = \frac{z}{R_x}\tilde{u}_0 + \int_0^z \left[-\tilde{m}\tilde{\phi}_{31} - \left(\frac{c_{44}Q}{c_{45}^2 - c_{44}c_{55}} \right)\tilde{\tau}_{xz(0)} \right] d\eta, \tag{A16}$$

$$\tilde{\phi}_{21}(z) = \frac{z}{R_y}\tilde{v}_0 + \int_0^z \left[-\tilde{n}\tilde{\phi}_{31} - \left(\frac{c_{55}Q}{c_{45}^2 - c_{44}c_{55}} \right)\tilde{\tau}_{yz(0)} \right] d\eta, \tag{A17}$$

$$\tilde{\phi}_{31}(z) = \int_0^z \left[\left(\frac{\tilde{m}\tilde{c}_{13}}{\gamma_\alpha} \right)\tilde{u}_{(0)} + \left(\frac{\tilde{n}\tilde{c}_{23}}{\gamma_\beta} \right)\tilde{v}_{(0)} - \left(\frac{\tilde{c}_{13}}{\gamma_\alpha R_x} + \frac{\tilde{c}_{23}}{\gamma_\beta R_y} \right)\tilde{w}_{(0)} \right] d\eta. \tag{A18}$$

At the ε^2-order level,

$$\tilde{u}_{(1)} = U_{1(mn)} - z\tilde{m}W_{1(mn)} + \tilde{\phi}_{11}, \tag{A19}$$

$$\tilde{v}_{(1)} = V_{1(mn)} - z\tilde{n}W_{1(mn)} + \tilde{\phi}_{21}, \tag{A20}$$

$$\tilde{w}_{(1)} = W_{1(mn)} + \tilde{\phi}_{31}, \tag{A21}$$

$$\tilde{\sigma}_{x(1)} = -\frac{\tilde{Q}_{11}\tilde{m}}{\gamma_\alpha}U_{1(mn)} - \frac{\tilde{Q}_{12}\tilde{n}}{\gamma_\beta}V_{1(mn)} + \left(\frac{\tilde{Q}_{11}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{12}}{\gamma_\beta R_y}\right)W_{1(mn)} + \tilde{c}_{13}\tilde{\sigma}_{z(0)}, \tag{A22}$$

$$\tilde{\sigma}_{y(1)} = -\frac{\tilde{Q}_{12}\tilde{m}}{\gamma_\alpha}U_{1(mn)} - \frac{\tilde{Q}_{22}\tilde{n}}{\gamma_\beta}V_{1(mn)} + \left(\frac{\tilde{Q}_{12}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{22}}{\gamma_\beta R_y}\right)W_{1(mn)} + \tilde{c}_{23}\tilde{\sigma}_{z(0)}, \tag{A23}$$

$$\tilde{\tau}_{xy(1)} = \frac{\tilde{Q}_{66}\tilde{m}}{\gamma_\beta}U_{1(mn)} + \frac{\tilde{Q}_{66}\tilde{n}}{\gamma_\alpha}V_{1(mn)}, \tag{A24}$$

$$\tilde{\tau}_{xz(1)} = \tilde{f}_{11} - \lambda\hat{f}_{11} + \int_{-1}^{z}\left[\left(\frac{\tilde{m}^2\tilde{Q}_{11}\gamma_\beta}{\gamma_\alpha} + \frac{\tilde{n}^2\tilde{Q}_{66}\gamma_\alpha}{\gamma_\beta}\right)\tilde{u}_{(1)} + \tilde{m}\tilde{n}(\tilde{Q}_{12}+\tilde{Q}_{66})\tilde{v}_{(1)} - \tilde{m}\left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha} + \frac{\tilde{Q}_{12}}{R_y}\right)\tilde{w}_{1(mn)} - \rho_1\omega^2\tilde{u}_{(1)}\right]\mathrm{d}\eta, \tag{A25}$$

$$\tilde{\tau}_{yz(1)} = \tilde{f}_{21} - \lambda\hat{f}_{21} + \int_{-1}^{z}\left[\tilde{m}\tilde{n}(\tilde{Q}_{12}+\tilde{Q}_{66})\tilde{u}_{(1)} + \left(\frac{\tilde{m}^2 Q_{66}\gamma_\beta}{\gamma_\alpha} + \frac{\tilde{n}^2\tilde{Q}_{22}\gamma_\alpha}{\gamma_\beta}\right)\tilde{v}_{(1)} - \tilde{n}\left(\frac{\tilde{Q}_{12}}{R_x} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta}\right)\tilde{w}_{1(mn)} - \rho_1\omega^2\tilde{v}_{(1)}\right]\mathrm{d}\eta, \tag{A26}$$

$$\tilde{\sigma}_{z(1)} = -\left(\frac{1}{R_x} + \frac{1}{R_y}\right)z\tilde{\sigma}_{z(0)} + \int_{-1}^{z}\left\{-\tilde{m}\left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha} + \frac{\tilde{Q}_{12}}{R_y}\right)\tilde{u}_{(1)} - \tilde{n}\left(\frac{\tilde{Q}_{12}}{R_x} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta}\right)\tilde{v}_{(1)} + \left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x^2\gamma_\alpha} + \frac{2\tilde{Q}_{12}}{R_xR_y} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y^2\gamma_\beta}\right)\tilde{w}_{(1)} + \tilde{m}\tilde{\tau}_{xz(1)} + \tilde{n}\tilde{\tau}_{yz(1)} + \frac{\tilde{m}\eta}{R_y}\tilde{\tau}_{xz(0)} + \frac{\tilde{n}\eta}{R_x}\tilde{\tau}_{yz(0)} + \left(\frac{\tilde{c}_{13}}{R_x} + \frac{\tilde{c}_{23}}{R_y}\right)\tilde{\sigma}_{z(0)} - \rho_2\omega^2\tilde{w}_{(1)} - 2\rho_2\omega\lambda\tilde{w}_{(0)}\right\}\mathrm{d}\eta. \tag{A27}$$